JOURNAL OF MATHEMATICAL PHYSICS

VOLUME 8, NUMBER 11 NOVEMBER 1967

Spin-s Spherical Harmonics and ð

J. N. GOLDBERG* AND A. J. MACFARLANE†
Syracuse University, Syracuse, New York
AND
E. T. NEWMAN‡
University of Pittsburgh, Pittsburgh, Pennsylvania
AND
F. ROHRLICH* AND E. C. G. SUDARSHAN†
Syracuse University, Syracuse, New York

(Received 26 September 1966)

Recent work on the Bondi–Metzner–Sachs group introduced a class of functions ${}_sY_{lm}(\theta, \phi)$ defined on the sphere and a related differential operator ð. In this paper the ${}_sY_{lm}$ are related to the representation matrices of the rotation group R_3 and the properties of ð are derived from its relationship to an angular-momentum raising operator. The relationship of the ${}_sT_{lm}(\theta, \phi)$ to the spherical harmonics of R_4 is also indicated. Finally using the relationship of the Lorentz group to the conformal group of the sphere, the behavior of the ${}_sT_{lm}$ under this latter group is shown to realize a representation of the Lorentz group.

1. INTRODUCTION

A RECENT paper by Newman and Penrose on the Bondi–Metzner–Sachs group[1] features a new differential operator,[2] symbolized by ð ("edth," the phonetic symbol for the hard "th"), and a related class of functions ${}_sY_{lm}(\theta, \phi)$, all defined on a sphere, in a central formal role. It is the purpose of the present paper to study ð and these generalized spherical functions and to relate them to more familiar structures.

In Sec. 2, we review previous work and give some further geometrical interpretation of thop as well as an illustration of the suitability of ð and the ${}_sY_{lm}(\theta, \phi)$, $s = 1, 0, -1$, in the manipulation of Maxwell's equations. In Sec. 3, we introduce and develop the formalism which allows one to view ð as a thinly disguised angular-momentum lowering operator and to relate the ${}_sY_{lm}(\theta, \phi)$ to the elements of the representation matrices of the rotation group R_3. This work was on the one hand motivated by inspection of the results reviewed in Sec. 2 and on the other hand allows a simple rederivation and ready extensions of such results. As an adjunct to this section, the relationship of ${}_sY_{lm}(\theta, \phi)$ to the spherical harmonics of R_4, i.e., those functions which carry the representations of R_4 defined on the unit sphere in four dimensions, is briefly indicated. In Sec. 4, we discuss the well-known relationship of the Lorentz group to the conformal group of the sphere and determine the behavior of the ${}_sY_{lm}$ under the conformal group, thereby realizing a representation of the Lorentz group of somewhat unusual appearance.

* This research was supported in part by the National Science Foundation.
† Research supported in part by the U.S. Atomic Energy Commission.
‡ Research supported in part by the U.S. Air Force Office of Scientific Research.

[1] E. Newman and R. Penrose, J. Math. Phys. 7, 863 (1966).
[2] The operator symbolized by ð has been referred to colloquially as "thop."

2. SUMMARY OF PREVIOUS WORK

In this section we discuss some of the previous work[1] on the differential operator ð and the spin-s spherical harmonics ${}_sY_{lm}$.

In three-dimensional Euclidean space with polar coordinates r, θ, ϕ, we introduce an orthonormal triad **a**, **b**, and **c** of vector fields. The vectors **a** and **b** are tangent to the sphere of radius r at each of its points while **c** is in the direction of the radius vector **r**. Of course **a** and **b** are only defined up to a rotation of angle ψ about **c**. It is very convenient to introduce in place of **a** and **b** the complex vector **m** and its complex conjugate $\bar{\mathbf{m}}$ by means of

$$\sqrt{2}\mathbf{m} = \mathbf{a} + i\mathbf{b}; \qquad (2.1)$$

then **m** is defined up to a phase factor, i.e., $\mathbf{m}' = e^{i\psi}\mathbf{m}$. A quantity η is now said to be of (integral) spin-weight s if, under (2.1), it transforms according to

$$\eta' = e^{is\psi}\eta. \qquad (2.2)$$

Examples of quantities of spin weights $s = 1, 0, -1$, respectively, are

$$\mathbf{A}\cdot\mathbf{m},\ \mathbf{A}\cdot\mathbf{c},\ \mathbf{A}\cdot\bar{\mathbf{m}},$$

where **A** is any vector. More generally, examples of quantities of spin-weight s are furnished by three-dimensional tensors of rank n contracted k_1, k_2, and k_3 times with **m**, **c**, and $\bar{\mathbf{m}}$, respectively, where $k_1 - k_3 = s$, $k_1 + k_2 + k_3 = n$. We adopt the convention that the real and imaginary parts of **m** point along the coordinate lines and hence transform according to (2.2) under coordinate transformations.

The differential operator ð, acting on a quantity η of spin-weight s, is defined by

$$ð\eta = -(\sin\theta)^s\left[\frac{\partial}{\partial\theta} + i\csc\theta\,\frac{\partial}{\partial\phi}\right](\sin\theta)^{-s}\eta. \qquad (2.3)$$

Since one has

$$(ð\eta)' = e^{i(s+1)\psi}(ð\eta), \qquad (2.4)$$

it is seen that ð has the important property of raising the spin weight by 1. Similarly if one defines $\bar{ð}$ by

$$\bar{ð}\eta = -(\sin\theta)^{-s}\left[\frac{\partial}{\partial\theta} - i\csc\theta\,\frac{\partial}{\partial\phi}\right](\sin\theta)^{s}\eta \qquad (2.3a)$$

with η here also a quantity of spin-weight s, one can see that $\bar{ð}$ lowers the spin weight by 1. Also one has

$$(\bar{ð}ð - ð\bar{ð})\eta = 2s\eta.$$

Of importance too is the effect of ð on ordinary spherical harmonics:

$$Y_{lm}(\theta,\phi), \quad -l \le m \le l, \quad l = 0, 1, 2, \cdots.$$

Indeed we can define spin-s spherical harmonics ${}_sY_{lm}$ for integral s, l, and m by

$$\begin{aligned}{}_sY_{lm}(\theta,\phi) &= [(l-s)!/(l+s)!]^{\frac{1}{2}}ð^sY_{lm}(\theta,\phi), \\ &\qquad\qquad 0 \le s \le l, \\ &= [(l+s)!/(l-s)!]^{\frac{1}{2}}(-)^s\bar{ð}^{-s}Y_{lm}(\theta,\phi), \\ &\qquad\qquad -l \le s \le 0. \end{aligned} \qquad (2.5)$$

The ${}_sY_{lm}$ (which are not defined for $|s| > l$) form a complete orthonormal set for each value of s; i.e., any spin-weight s function can be expanded in a series in ${}_sY_{lm}$. The spin-s spherical harmonics have the further properties:

(i) $\quad {}_s\bar{Y}_{lm} = (-)^{m+s}{}_{-s}Y_{lm}, \qquad (2.6)$

(ii) $\quad ð\,{}_sY_{lm} = [(l-s)(l+s+1)]^{\frac{1}{2}}{}_{s+1}Y_{lm}, \qquad (2.7a)$

(iii) $\quad \bar{ð}\,{}_sY_{lm} = -[(l+s)(l-s+1)]^{\frac{1}{2}}{}_{s-1}Y_{lm}, \qquad (2.7b)$

(iv) $\quad \bar{ð}ð\,{}_sY_{lm} = -(l-s)(l+s+1)\,{}_sY_{lm}. \qquad (2.8)$

Thus ð and $\bar{ð}$ act as raising and lowering operators on the "quantum number" s, and the ${}_sY_{lm}$ are eigenfunctions of $\bar{ð}ð$.

For many computations, a more convenient coordinate system for the sphere is the set of complex stereographic coordinates $(\zeta, \bar{\zeta})$ which are introduced by

$$\zeta = e^{i\phi}\cot\tfrac{1}{2}\theta. \qquad (2.9)$$

ð and $\bar{ð}$ become

$$\begin{aligned} ð\eta &= 2P^{1-s}[\partial(P^s\eta)/\partial\zeta], \\ \bar{ð}\eta &= 2P^{1+s}[\partial(P^{-s}\eta)/\partial\bar{\zeta}], \end{aligned} \qquad (2.10)$$

with $P = \frac{1}{2}(1 + \zeta\bar{\zeta})$. In the $(\zeta, \bar{\zeta})$ system, the spin-s spherical harmonics take the form

$$\begin{aligned}{}_sY_{lm} &= \frac{a_{lm}}{[(l-s)!\,(l+s)!]^{\frac{1}{2}}}(1+\zeta\bar{\zeta})^{-l} \\ &\quad\times \sum_p \binom{l-s}{p}\binom{l+s}{p+s-m}\zeta^p(-\bar{\zeta})^{p+s-m}, \end{aligned} \qquad (2.11)$$

with

$$a_{lm} = (-)^{l-m}[(l+m)!\,(l-m)!\,(2l+1)/4\pi]^{\frac{1}{2}}. \qquad (2.12)$$

Expression (2.11) applies also to "spinor harmonics" for which l, m, and s are all half-odd integers.

ð can be related to covariant differentiation in the following manner: using coordinates on the sphere such that the metric takes the form[3]

$$ds^2 = P^{-2}\,d\zeta\,d\bar{\zeta},$$

we introduce two complex vectors $m^\alpha = \sqrt{2}P\delta^\alpha_\zeta$, $\bar{m}^\alpha = \sqrt{2}\delta^\alpha_{\bar{\zeta}}$, $\alpha = \zeta, \bar{\zeta}$. From a spin-weight s quantity η, we can define a totally symmetric and trace-free

[3] The function P and the coordinates ζ need not be the ones used in Eq. (2.10); as a matter of fact the surface need not even be a sphere.

tensor of rank s,

$$\eta_{(\alpha\cdots\beta)} = \eta\bar{m}_\alpha\cdots\bar{m}_\beta + \bar{\eta}m_\alpha\cdots m_\beta,$$

with the inverse relations

$$\eta = \eta_{(\alpha\cdots\beta)}m^\alpha\cdots m^\beta;\ \bar{\eta} = \eta_{(\alpha\cdots\beta)}\bar{m}^\alpha\cdots\bar{m}^\beta.$$

It is now easy to prove

$$ð\eta = \sqrt{2}\eta_{(\alpha\cdots\beta);\gamma}m^\alpha\cdots m^\beta m^\gamma. \tag{2.13}$$

As a simple example illustrating the use of ð and the ${}_sY_{lm}$, we consider the Maxwell equations

$$\begin{aligned}\nabla\cdot(\mathbf{E}+i\mathbf{B}) &= 0,\\ \nabla\wedge(\mathbf{E}+i\mathbf{B}) - i(\partial/\partial t)(\mathbf{E}+i\mathbf{B}) &= 0.\end{aligned} \tag{2.14}$$

The quantities[4]

$$\begin{aligned}G_+ &= (\mathbf{E}+i\mathbf{B})\cdot\mathbf{m},\\ G_0 &= (\mathbf{E}+i\mathbf{B})\cdot\mathbf{c},\\ G_- &= (\mathbf{E}+i\mathbf{B})\cdot\bar{\mathbf{m}}\end{aligned} \tag{2.15}$$

of spin weight 1, 0, and -1, respectively, can be shown from (2.14) to satisfy the equations

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)rG_+ - ð\bar{ð}_r G_+ = 0, \tag{2.16a}$$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)r^2G_0 - \bar{ð}ð_r G_0 = 0, \tag{2.16b}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)rG_- - ðð_r G_- = 0, \tag{2.16c}$$

in which the quantities G_+, G_0, G_- are already uncoupled. If we assume solutions of these equations of the form

$$\begin{aligned}rG_+ &= F_+(\mathbf{r},t)_1Y_{lm}(\theta\phi),\\ r^2G_0 &= F_0(\mathbf{r},t)_0Y_{lm}(\theta\phi),\\ rG_- &= F_-(\mathbf{r},t)_{-1}Y_{lm}(\theta\phi),\end{aligned} \tag{2.17}$$

it is seen from Eqs. (2.7) and (2.8) that

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)F_+ + (l-1)(l+2)F_+ &= 0,\\ \left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}\right)F_0 + \frac{1}{r^2}l(l+1)F_0 &= 0,\\ \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)F_- + (l-1)(l+2)F_- &= 0,\end{aligned} \tag{2.18}$$

the dependence on angular variables having canceled out. These latter equations can be solved by a variety of standard techniques, though it is not our purpose to go into this question here.

The main point to be made is that Maxwell's equations or more generally vector equations can be simply solved in terms of the ${}_sY_{lm}$ instead of the cumbersome apparatus of the vector spherical harmonics.[5]

3. RELATIONSHIP TO R_3 AND R_4

In this section, we identify the functions ${}_sY_{lm}$ with the elements of the matrices of the representation D^l of the ordinary rotation group R_3, and relate ð to an ordinary angular-momentum raising operator. We thereby obtain the principal properties of the ${}_sY_{lm}$ and ð as transcriptions of results familiar in the theory of angular momentum.

We proceed first to the above mentioned identification of the ${}_sY_{lm}$. For our purpose it is convenient to have an explicit definition of ${}_sY_{lm}(\theta,\phi)$ rather than the expression in terms of stereographic coordinates given in Eq. (2.11). By direct substitution of (2.9) we obtain[6]

$$\begin{aligned}{}_sY_{lm}(\theta,\phi) &= \left[\frac{(l+m)!\,(l-m)!}{(l+s)!\,(l-s)!}\frac{(2l+1)}{4\pi}\right]^{\frac{1}{2}}(\sin\theta/2)^{2l}\\ &\times\sum_r\binom{l-s}{r}\binom{l+s}{r+s-m}(-)^{l-r-s}e^{im\phi}(\cot\theta/2)^{2r+s-m}.\end{aligned} \tag{3.1}$$

Now we give[7] careful definitions of and appropriate explicit formulas for the elements of the matrix D^l, of the representation of R_3 associated with total angular momentum l. If a spatial rotation R of angle ω about a unit vector $\mathbf{n}$ is given by

$$\begin{aligned}x^k \to x'^k &= R^{kl}x^l,\\ R^{kl} &= \delta^{kl}\cos\omega + n^kn^l(1-\cos\omega) - \epsilon^{klm}\sin\omega,\end{aligned} \tag{3.2}$$

then the matrix D^l may be defined by its action on spherical harmonics

$$\begin{aligned}Y_{lm}(\hat{\mathbf{x}}) &= \langle\hat{\mathbf{x}}\mid lm\rangle \to Y_{lm}(\hat{\mathbf{x}}'),\\ \hat{\mathbf{x}} &= (\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta),\\ Y_{lm}(\hat{\mathbf{x}}') &= \sum_{m'}Y_{lm'}(\mathbf{x})D^l_{m'm}(R^{-1}).\end{aligned} \tag{3.3}$$

[4] G_+, G_0, G_- have been referred to elsewhere as ϕ_0, ϕ_1, ϕ_2. See, e.g., E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[5] It has recently been pointed out to us that the functions ${}_sY_{lm}$ have already been introduced, though by very different techniques. For this alternate method, and its detailed application to Maxwell theory, see I. M. Gel'fand, R. A. Minlos, and Z. Ya Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (The Macmillan Company, New York, 1963).

[6] In this passage from Eq. (2.11) defining ${}_sY_{lm}(\zeta,\bar{\zeta})$ to Eq. (3.1) one should not only insert the definition (2.9) but also introduce an additional phase factor $e^{is\phi}$ to account for the rotation of the vectors associated with the change of coordinates $(\zeta,\bar{\zeta})$ to (θ,ϕ).

[7] The necessity for the detail of the discussion here stems from the fact that we could not simply refer to one of the few completely consistent treatments of the theory of the rotation group available in the literature, without extensive modification of the notation employed in Ref. 4 and related papers.

If we define a rotation $R(\alpha, \beta, \gamma)$ of Euler angles α, β, γ as being composed of[8] γ about OZ followed by β about OY and then α about OZ we have

$$\begin{aligned} D^l_{m'm}(\alpha\beta\gamma) &\equiv D^l_{m'm}(R(\alpha\beta\gamma)^{-1}) \\ &= e^{im'\gamma} d^l_{m'm}(\beta) e^{im\alpha}. \end{aligned} \tag{3.4}$$

Following Wigner,[9] in principle if not in detail, we employ the relationship of R_3 to SU_2 in order to give an explicit formula for $D^l_{m'm}(\alpha\beta\gamma)$. If the element A of SU_2 acts on a two-component spinor $w = \binom{u}{v}$, where

$$u = e^{i\phi/2} \cos \tfrac{1}{2}\theta, \quad v = e^{-i\phi/2} \sin \tfrac{1}{2}\theta, \tag{3.5}$$

so that $u/v = \zeta$, according to $w \to w' = Aw$, then the correspondence of $A \in SU_2$ to $R \in R_3$ can be given in the form[10]

$$\begin{aligned} R^{kl} &= \tfrac{1}{2} \operatorname{Tr} (\sigma^k A \sigma^l A^\dagger), \\ A &= \pm(1 + \sigma^k \sigma^l R^{kl})/[4(1 + \operatorname{Tr} R)]^{\frac{1}{2}}, \end{aligned} \tag{3.6}$$

which allows us to obtain the image $A(\alpha\beta\gamma)$ of $R(\alpha\beta\gamma)$ in the form

$$A(\alpha\beta\gamma) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a = e^{-\frac{1}{2}i(\alpha+\gamma)} \cos \tfrac{1}{2}\beta, \quad b = e^{-\frac{1}{2}i(\alpha-\gamma)} \sin \tfrac{1}{2}\beta. \tag{3.7}$$

Now defining

$$\phi_{jm}(u, v) = \frac{u^{j+m} v^{j-m}}{[(j+m)!\,(j-m)!]^{\frac{1}{2}}}$$

as usual we can, in agreement with Eq. (3.5), write

$$\begin{aligned} &\phi_{jm}(u, v) \to \phi_{jm}(u', v'), \\ &\phi_{jm}(u', v') = \sum_{m'} \phi_{jm'}(u, v) D^j_{m'm}(A(\alpha\beta\gamma)^{-1}), \end{aligned} \tag{3.8}$$

and with a little algebra obtain

$$\begin{aligned} &D^j_{m'm}(\alpha\beta\gamma) \\ &\equiv D^j_{m'm}(A(\alpha\beta\gamma)^{-1}) \\ &= \left[\frac{(j+m)!\,(j-m)!}{(j+m')!\,(j-m')!}\right]^{\frac{1}{2}} \sum_r \binom{j+m'}{r} \binom{j-m'}{r-m-m'} \\ &\qquad \times \bar{a}^r \bar{b}^{j+m-r} (-b)^{j+m'-r} a^{r-m-m'} \\ &= \left[\frac{(j+m)!\,(j-m)!}{(j+m')!\,(j-m')!}\right]^{\frac{1}{2}} (\sin \tfrac{1}{2}\beta)^{2j} \\ &\quad \times \sum_r \binom{j+m'}{r} \binom{j-m'}{r-m-m'} (-)^{j+m'-r} \\ &\quad \times e^{im\alpha} (\cot \tfrac{1}{2}\beta)^{2r-m-m'} e^{im'\gamma}. \end{aligned} \tag{3.9}$$

[8] This procedure is clearly equivalent to the more usual one of a rotation α around OZ, followed by β around OY' and finally γ around OZ''.

[9] E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959).

[10] It follows now that under $w \to w' = \bar{A}w$, $W = \tilde{w}\sigma\bar{w}$ $(= \hat{\mathbf{x}})$ has the transformation law $W^k = W'^k = R^{kl}W^l$ as consistency of course requires.

We may now insert $\alpha = \phi$, $\beta = \theta$, $j = l$, $m' = -s$ into Eq. (3.9) and, by comparison with Eq. (3.3) obtain

$${}_sY_{lm}(\theta\phi)e^{-is\gamma} = [(2l+1)/4\pi] D^l_{-sm}(\phi\theta\gamma), \tag{3.10}$$

so that for $\gamma = 0$ we can make the promised identification

$${}_sY_{lm}(\theta\phi) = [(2l+1)/4\pi]^{\frac{1}{2}} D^l_{-sm}(\phi\theta 0). \tag{3.11}$$

Note that for $s = 0$, we have

$${}_0Y_{lm}(\theta\phi) = [(2l+1)/4\pi]^{\frac{1}{2}} D^l_{0m}(\phi\theta 0) = Y_{lm}(\theta\phi),$$

so that the spin-s spherical harmonics with spin weight $s = 0$ are exactly the ordinary spherical harmonics. It may also be noted that our procedure extends the definition of spin-s spherical harmonics to the case of s half-integral.

Now the functions $D^l_{m'm}(\alpha\beta\gamma)$ provide[11] a complete orthonormal basis for functions defined on R_3, so that orthogonality and completeness relations for ${}_sY_{lm}(\theta, \phi)$ follow easily. The orthogonality relations

$$\begin{aligned} \int_0^{2\pi} d\alpha \int_{-l}^{1} d\cos\beta \int_0^{2\pi} d\gamma\, \bar{D}^l_{-sm}(\alpha\beta\gamma) D^{l'}_{-s'm'}(\alpha\beta\gamma) \\ = [8\pi^2/(2l+1)]\delta_{ll'}\delta_{mm'}\delta_{ss'} \end{aligned} \tag{3.12}$$

translate, by use of Eq. (3.10) and relabeling, into

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, {}_s\bar{Y}_{lm}(\theta\phi)\, {}_sY_{l'm'}(\theta\phi) = \delta_{ll'}\delta_{mm'}. \tag{3.13}$$

It is noteworthy that we obtain in this way only an orthogonality relation involving spin-s spherical harmonics of the same spin weight. Orthogonality of the D^l_{-sm} with respect to s in Eq. (3.12) is of course associated with the variable γ which is absent in Eq. (3.13). Also from the completeness relation

$$\begin{aligned} &\sum_{lmm'} \bar{D}^l_{m'm}(\alpha\beta\gamma) D^l_{m'm}(\alpha'\beta'\gamma') \\ &\quad = 8\pi^2/(2l+1)\delta(\alpha-\alpha')\delta(\cos\beta - \cos\beta')\delta(\gamma-\gamma'), \end{aligned} \tag{3.14}$$

we can prove, by evaluating

$$\int_0^{2\pi} d\gamma e^{-is\gamma}(\cdots)$$

(where s is any integer) on both sides, that we have a completeness relation

$$\sum_{lm} {}_s\bar{Y}_{lm}(\theta\phi)\, {}_sY_{lm}(\theta'\phi') = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta'), \tag{3.15}$$

for each integral value of s. Thus for each integral s the function ${}_sY_{lm}(\theta\phi)$ form according to Eqs. (3.13) and (3.15) a complete orthonormal set of functions on the unit sphere with respect to which any function of spin-weight s defined on the unit sphere can be expanded.

[11] R. Penrose, Proc. Cambridge Phil. Soc. **55**, 137 (1959).

We turn now to the use of Eqs. (3.10) and (3.11) to motivate the association of ð with an angular-momentum raising operator. We set out from the observation, familiar from the theory of the symmetric top, that if one defines operators L_z, $L_\pm$,

$$L_z = -i\frac{\partial}{\partial\alpha},$$
$$L_\pm = \pm e^{\pm i\alpha}\left(\frac{\partial}{\partial\beta} \pm i\cot\beta\frac{\partial}{\partial\alpha} \pm i\csc\beta\frac{\partial}{\partial\gamma}\right) \quad (3.16)$$

which obey the commutation relations

$$[L_z, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = 2L_z$$

of angular momentum, then for each allowed value of s, $D^l_{-sm}(\alpha\beta\gamma)$ behaves like an eigenvector $|lm\rangle$, i.e.,

$$\mathbf{L}^2 D^l_{-sm} = l(l+1)D^l_{-sm},$$
$$L_z D^l_{-sm} = m D^l_{-sm},$$
$$L_\pm D^l_{-sm} = [(l \mp m)(l \pm m + 1)]\; D^l_{-sm\pm1}. \quad (3.17)$$

We do not relate L_+ to ð, of course, but instead define a second angular-momentum operator $\mathbf{K}$, which commutes with $\mathbf{L}$, and with respect to which D^l_{-sm} behaves like an eigenvector $|ls\rangle$ for each allowed value of m. The way to define $\mathbf{K}$ follows easily from the symmetry of $D^l_{-sm}(\alpha, \beta, \gamma)$ with respect to m, α on the one hand and $s - \gamma$, on the other. Thus, we define

$$K_z = i\frac{\partial}{\partial\gamma},$$
$$K_\pm = \pm e^{\pm i\gamma}\left(\frac{\partial}{\partial\beta} \pm i\cot\beta\frac{\partial}{\partial\gamma} \pm i\csc\beta\frac{\partial}{\partial\alpha}\right) \quad (3.18)$$

and deduce

$$[K_z, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = 2K_z,$$
$$[\mathbf{L}, \mathbf{K}] = 0,$$

and

$$\mathbf{K}^2 D^l_{-sm} = l(l+1)\; D_l^{-sm},$$
$$K_z D^l_{-sm} = s D^l_{-sm},$$
$$K_\pm D^l_{-sm} = [(l \mp s)(l \pm s + 1)]^{\frac{1}{2}} D^l_{-(s\pm1)m}. \quad (3.19)$$

We are now in a position to make explicit the relationship of K_+ to ð. When acting on D^l_{-sm} the operator K_+ can be written in the form

$$K_+ = e^{-i\gamma}\left(\frac{\partial}{\partial\beta} - is\cot\beta + i\csc\beta\frac{\partial}{\partial\alpha}\right)$$
$$= e^{-i\gamma}(\sin\beta)^s\left(\frac{\partial}{\partial\beta} + i\csc\beta\frac{\partial}{\partial\alpha}\right)(\sin\beta)^{-s}, \quad (3.20)$$

so that

$$[K_+ D^l_{-sm}]_{\alpha=\phi,\beta=\theta,\gamma=0} = ð D^l_{-sm}\,|\phi\theta 0) \quad (3.21)$$

follows in accordance with Eq. (3.1). Thus K_+ is the differential operator to which the operator ð is more closely related. The reason that ð is not defined as a differential operator by Newman and Penrose stems from the fact that they work only with ${}_sY_{lm}(\theta\phi) \sim D^l_{-sm}(\phi\theta 0)$ rather than $D^l_{-sm}(\phi\theta\gamma)$, i.e., from the nonappearance of the variable γ. Of course this in turn results from the fact that such a variable is not needed by them on physical grounds. However, the properties of ð follow very easily from its relation to K_+. For example, from (19), (21), and (10) we get directly

$$ð_s Y_{lm}(\theta\phi) = [(l-s)(l+s+1)]^{\frac{1}{2}}{}_{s+1}Y_{lm}(\theta\phi), \quad (3.22)$$

which is Eq. (3.22) of the paper by Newman and Penrose. Of course, it was results like this one which initially suggested the relationship of s to a magnetic quantum number and motivated the identifications of ð with an angular-momentum operator.

Finally, it may be worthwhile here to point out the relationship of ð to representations of R_4 defined on the unit 4-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. It is well known that the generators of infinitesimal rotations of R_4 can be defined according to

$$M_{kl} = -i(x_k\partial_l - x_l\partial_k), \quad 1 \le k, \quad l \le 4,$$

and replaced by a pair of commuting angular momentum operators $\mathfrak{L}$, $\mathfrak{K}$:

$$\mathfrak{L}_1 = M_{23} + M_{14}, \quad \mathfrak{L}_2 = M_{31} + M_{24},$$
$$\mathfrak{L}_3 = M_{12} + M_{34},$$
$$\mathfrak{K}_1 = M_{23} - M_{14}, \quad \mathfrak{K}_2 = M_{31} - M_{24},$$
$$\mathfrak{K}_3 = M_{12} - M_{34}.$$

Now, in view of the consequence $\mathfrak{L}^2 = \mathfrak{K}^2$ of these definitions, only the subset (l, k) of representations of R_4 with $l = k = 0, \frac{1}{2}, 1, \cdots$ can be defined on the unit 4-sphere with these "standard" definitions of the six infinitesimal generators. However, only these representations arise in the previous discussion. We can explicitly make contact with the formalism of the previous paragraph by introducing polar coordinates according to

$$x_1 = \sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}(\alpha - \gamma),$$
$$x_2 = \cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}(\alpha + \gamma),$$
$$x_3 = \sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}(\alpha - \gamma),$$
$$x_4 = \cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}(\alpha + \gamma), \quad (3.23)$$

for then $\mathfrak{L} = \mathbf{L}$, $\mathfrak{K} = \mathbf{K}$ follow. Alternatively we could remark that the $D^j_{m'm}(A^{-1})$ form a complete orthonormal basis for functions defined 'on $SU(2)$.' Explicitly this latter term refers to functions of a, b such that

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

belongs to SU(2), or simply such that $|a|^2 + |b|^2 = 1$. Now from (7) and (23), we have

$$a = x_2 - ix_4, \quad b = x_1 - ix_3,$$

so that functions of a, b such that

$$|a|^2 + |b|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

can be read as functions defined on the unit 4-sphere. This remark is of course what underlies the identification of the $D^j_{m'm}$ with the basis of the representation (j,j) or R_4.

It is perhaps worth emphasizing that the ${}_sY_{lm}(\theta\phi)$ or the D^l_{-sm} play two very different roles being on the one hand closely related to matrix elements of the representation matrices of O_3 and on the other hand closely related to bases functions of certain representations of O_4.

4. THE LORENTZ TRANSFORMATION AND SPIN-s SPHERICAL HARMONICS

A. Conformal Mappings

Up to this time the discussion of the spin-s spherical harmonics has been based on their relationship to the rotation group. The rigid rotations are a three-parameter group of isometric mappings of the unit sphere onto itself. Thus

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 = d\theta'^2 + \sin^2\theta'\, d\phi'^2 \quad (4.1)$$

if the mapping $\{\theta, \phi\} \to \{\theta', \phi'\}$ is a rigid rotation. In order to relate the spin-s spherical functions to the Lorentz group it is necessary to enlarge this group of homeomorphic mappings of the 2-sphere. The mapping $\{\theta, \phi\} \to \{\theta', \phi'\}$ is conformal if

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2\theta\, d\phi^2 \\ &= K^2(\theta', \phi')(d\theta'^2 + \sin^2\theta'\, d\phi'^2). \end{aligned} \quad (4.2)$$

Clearly the rigid rotations form that subgroup of the conformal transformations for which the conformal factor $K^2 = 1$. The conformal group, which preserves the angle between two curves and its direction, can be shown to be a six-parameter Lie group which is isomorphic to the proper homogeneous Lorentz group.[11,12] The result can be easily derived and as it introduces the notation we wish to use in our discussion of spin-s spherical functions, we give the proof here.

In terms of the stereographic coordinates $\zeta = e^{i\phi}\cot\theta/2$ which were introduced in Sec. 2, the metric on the unit sphere has the form

$$ds^2 = 4(1 + \zeta\bar{\zeta})^{-2}\, d\zeta\, d\bar{\zeta}. \quad (4.3)$$

The complex coordinate ζ defines a point in the complex plane. Therefore, the conformal transformations of the complex plane will induce the conformal transformations of the unit sphere onto itself. The only transformations with a simple pole and a simple zero at the new north and south poles, respectively, are given by the Möbius transformation

$$\zeta' = (\alpha\zeta + \beta)/(\gamma\zeta + \delta); \quad \alpha\delta - \beta\gamma = 1. \quad (4.4)$$

Applying this transformation to Eq. (4.3) we find

$$ds^2 = K^2 = K^2[4(1 + \zeta'\bar{\zeta}')^{-2}]\, d\zeta'\, d\bar{\zeta}', \quad (4.5)$$

$$K = \frac{(\alpha\zeta + \beta)(\bar{\alpha}\bar{\zeta} + \bar{\beta}) + (\gamma\zeta + \delta)(\bar{\gamma}\bar{\zeta} + \bar{\delta})}{1 + \zeta\bar{\zeta}}. \quad (4.6)$$

The complex constants α, β, γ, and δ together with the restriction indicated in Eq. (4.4) represent six real parameters.

To show the isomorphism of Eq. (4.3) with the proper homogeneous Lorentz group, we introduce a two-dimensional complex linear vector space. Let u_1 and u_2 be the components of a vector in this space. To each transformation (4) there corresponds a transformation of $SL(2)$ as follows:

$$u_1' = \alpha u_1 + \beta u_2, \quad u_2' = \gamma u_1 + \delta u_2, \quad (4.7)$$

as can be seen by the identification $\zeta = u_1/u_2$. $SL(2)$ furnishes a double covering of the conformal transformation exactly as it furnishes a double covering of the proper homogeneous Lorentz group. Thus the required isomorphism is established.

B. The Irreducible Representations $\mathfrak{D}^{(j_1)(j_2)}$

If ξ and η are two independent basis vectors in the two-dimensional spinor space [the space of vectors (u_1, u_2) which satisfy the transformation law (4.7)], then a basis for the linear vector space defining the irreducible representation of the Lorentz group denoted by[12] $\mathfrak{D}^{(j_1)(j_2)}$ is given by

$$\begin{gathered} (\xi^{2j_1-m_1}\eta^{m_1})(\bar{\xi}^{2j_2-m_2}\bar{\eta}^{m_2}), \\ 0 \le m_1 \le 2j_1, \quad 0 \le m_2 \le 2j_2. \end{gathered} \quad (4.8)$$

The parentheses indicate complete symmetrization of the factors. This linear vector space is $(2j_1 + 1)(2j_2 + 1)$ dimensional. Therefore, an arbitrary vector in this space is determined by $(2j_1 + 1)(2j_2 + 1) \times$ numbers a_{m_1,m_2}. The transformation (4) which maps (u_1, u_2) into (u_1', u_2') induces a corresponding mapping of the components a_{m_1,m_2} into components a'_{m_1,m_2}. By considering the transformation of the quantities

$$\begin{aligned} &(u_1')^{2j_1-m_1}(u_2')^{m_1}(\bar{u}_1')^{2j_2-m_2}(\bar{u}_2')^{m_2} \\ &= (\alpha u_1 + \beta u_2)^{2j_1-m_1}(\gamma u_1 + \delta u_2)^{m_1} \\ &\quad (\bar{\alpha}\bar{u}_1 + \bar{\beta}\bar{u}_2)^{2j_2-m_2}(\bar{\gamma}u_1 + \bar{\delta}\bar{u}_2)^{m_2} \\ &= \sum_{n_1=0}^{2j_1}\sum_{n_2=0}^{2j_2} A^{(j_1)(j_2)}_{m_1m_2;n_1n_2} u_1^{2j_1-n_1}u_2^{n_1}\bar{u}_1^{2j_2-n_2}\bar{u}_2^{n_2}, \end{aligned} \quad (4.9)$$

we establish the transformation

$$a'_{m_1m_2} = \sum_{n_1=0}^{2j_1}\sum_{n_2=0}^{2j_2} A^{(j_1)(j_2)}_{m_1m_2;n_1n_2} a_{n_1n_2}. \quad (4.10)$$

[12] R. K. Sachs, Phys. Rev. **128**, 385 (1962).

[13] See, for example, P. Roman, *Theory of Elementary Particle* (North-Holland Publishing Company, Amsterdam, 1960).

C. The Transformation of the Spin-s Spherical Harmonics

Consider the set of functions

$$ {}_sZ^L_{m_1m_2} = (1+\zeta\bar\zeta)^{-L}\zeta^{L-s-m_1}\bar\zeta^{L+s-m_2}, $$
$$ |s| \le L, \quad 0 \le m_1 \le L-s, \quad 0 \le m_2 \le L+s. \tag{4.11} $$

Applying the transformation (4.4), we get for the transformed set

$$ {}_sZ'^L_{m_1m_2} = [e^{is\lambda}/K^L(1+\zeta\bar\zeta)^L]\{(\alpha\zeta+\beta)^{L-m_1}(\alpha\zeta+\delta)^{m_1} \times (\bar\alpha\bar\zeta+\bar\beta)^{L+s-m_2}(\bar\gamma\bar\zeta+\bar\delta)^{m_2}\} \tag{4.12} $$

with

$$ \frac{1}{1+\zeta'\bar\zeta'} = \frac{(\gamma\zeta+\delta)(\bar\gamma\bar\zeta+\bar\delta)}{K(1+\zeta\bar\zeta)}, $$
$$ e^{i\lambda} = \frac{\gamma\zeta+\delta}{\bar\gamma\bar\zeta+\bar\delta}, $$

and K given by Eq. (4.6). Comparing (4.12) with Eq. (4.9) and (4.10), we find that

$$ {}_sZ'^L_{m_1m_2} = K^{-L}e^{is\lambda}\sum_{n_1=0}^{L-s}\sum_{n_2=0}^{L+s} A^{[\frac12(L-s)][\frac12(L+s)]}_{m_1m_2;n_1n_2}\,{}_sZ^L_{n_1n_2}. $$

Therefore, up to the conformal factor $K^{-L}e^{is\lambda}$, the functions ${}_sZ^L_{m_1m_2}$ transform under the $\mathfrak{D}^{[\frac12(L-s)][\frac12(L+s)]}$ irreducible representation of the Lorentz group.

Clearly these functions do not form an orthonormal set of functions on the sphere for fixed s. Indeed, for all $L > |s|$ they form a redundant set of functions for definite spin-weight s. However, the spin-s spherical harmonics ${}_sY_{lm}$ do form an orthonormal set for fixed s. It is easy to show that for $l \le L$ the ${}_sY_{lm}$ are given uniquely by the ${}_sZ^L_{m'm}$:

$$ {}_sY_{lm} = \sum_{m_1=0}^{L-s}\sum_{m_2=0}^{L+s} {}_sB^{L\ m_1m_2}_{lm}\,{}_sZ^L_{m_1m_2}, \tag{4.13} $$
$$ s \le l \le L, \quad |m| \le l; $$

$$ {}_sB^{L\ m_1m_2}_{lm} = \frac{a_{lm}}{[(l-s)!\,(l+s)!]^{\frac12}}\sum_{p=0}^{p_m}(-1)^{p+s-m}\binom{l-s}{p} \times \binom{l+s}{p+s-m}\binom{L-l}{L-s-m_1-p} \times \delta_{m_2,m_1+s+m}, \tag{4.14} $$

$$ p_m = \min\{L-s-m_1,\ l-s,\ l+m\} \tag{4.14a} $$

and the a_{lm} are the constants defined in Eq. (2.11).

For fixed s and L the coefficients ${}_sB^{L\ m_1m_2}_{lm}$ form a nonsingular

$$ (L-s+1)(L+s+1) \times (L-s+1)(L+s+1) $$

matrix $[(l,m),(m_1,m_2)]$ connecting the ${}_sZ^L_{m_1m_2}$ to the ${}_sY_{lm}$. Since the ${}_sZ^L_{m_1m_2}$ transform under the

$$ \mathfrak{D}^{[\frac12(L-s)][\frac12(L+s)]} $$

representation of the Lorentz group up to the factor $K^{-L}e^{il\lambda}$, it follows that the ${}_sY_{lm}(|s| \le l \le L$ and $|m| \le l)$ transform under an equivalent representation up to the same factor.

The above results hold both for L and s integral, or half-integral.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Invariant Transformations and Newman–Penrose Constants*

JOSHUA N. GOLDBERG
Physics Department, Syracuse University, Syracuse, New York

(Received 29 November 1966)

The relationship between constants of the motion and invariant transformations is discussed. Particular emphasis is placed on strong conservation laws which are of the form $\mathfrak{U}^{\mu\nu}_{\lambda\ ,\mu\nu} \equiv 0$. The existence of such a law leads to constants of the motion which are surface integrals and therefore generally do not generate an invariant transformation. However, when there is an associated weak conservation law, such that $t^{\mu}{}_{,\mu} \equiv -\delta y_A L^A$ ($L^A = 0$ are the field equations and δy_A the invariant change in field variables), a nontrivial invariant transformation exists. These results are applied to the discussion of the Newman–Penrose constants for the electromagnetic field. The conclusion arrived at is $\delta y_A = 0$, which suggests that the invariant transformation generated by the Newman–Penrose constants is trivial.

1. INTRODUCTION

A NEW set of conserved quantities for free massless fields has been discovered by Newman and Penrose.[1] For linear field theories, an infinite hierarchy of such conserved quantities can be defined. However, the nonlinear field equations of general relativity permit only the first set of these constants to exist and then only in asymptotically flat space–times. It is well known[2] that constants of the motion generate an invariant transformation, and conversely, that the

* This research was supported in part by the Aerospace Research Laboratories.

[1] E. T. Newman and R. Penrose, Phys. Rev. Letters **15**, 231 (1965).

[2] For example, see P. G. Bergmann, *Encyclopedia of Physics*, Vol. **54** (Springer-Verlag, Berlin, S. Flugge, Ed., 1962); A Trautman, "Conservation Laws," article in *Gravitation*, L. Witten, Ed. (John Wiley & Sons, Inc., New York, 1962).

C. The Transformation of the Spin-s Spherical Harmonics

Consider the set of functions

$$
{}_sZ^L_{m_1m_2} = (1+\zeta\bar{\zeta})^{-L}\zeta^{L-s-m_1}\bar{\zeta}^{L+s-m_2},
$$

$$
|s| \le L, \quad 0 \le m_1 \le L-s, \quad 0 \le m_2 \le L+s. \tag{4.11}
$$

Applying the transformation (4.4), we get for the transformed set

$$
{}_sZ'^L_{m_1m_2} = [e^{is\lambda}/K^L(1+\zeta\bar{\zeta})^L]\{(\alpha\zeta+\beta)^{L-m_1}(\alpha\zeta+\delta)^{m_1} \times (\bar{\alpha}\bar{\zeta}+\bar{\beta})^{L+s-m_2}(\bar{\gamma}\bar{\zeta}+\bar{\delta})^{m_2}\} \tag{4.12}
$$

with

$$
\frac{1}{1+\zeta'\bar{\zeta}'} = \frac{(\gamma\zeta+\delta)(\bar{\gamma}\bar{\zeta}+\bar{\delta})}{K(1+\zeta\bar{\zeta})},
$$

$$
e^{i\lambda} = \frac{\gamma\zeta+\delta}{\bar{\gamma}\bar{\zeta}+\bar{\delta}},
$$

and K given by Eq. (4.6). Comparing (4.12) with Eq. (4.9) and (4.10), we find that

$$
{}_sZ'^L_{m_1m_2} = K^{-L}e^{is\lambda}\sum_{n_1=0}^{L-s}\sum_{n_2=0}^{L+s} A^{[\frac{1}{2}(L-s)][\frac{1}{2}(L+s)]}_{m_1m_2;n_1n_2}\,{}_sZ^L_{n_1n_2}.
$$

Therefore, up to the conformal factor $K^{-L}e^{is\lambda}$, the functions ${}_sZ^L_{m_1m_2}$ transform under the $\mathfrak{D}^{[\frac{1}{2}(L-s)][\frac{1}{2}(L+s)]}$ irreducible representation of the Lorentz group.

Clearly these functions do not form an orthonormal set of functions on the sphere for fixed s. Indeed, for all $L > |s|$ they form a redundant set of functions for definite spin-weight s. However, the spin-s spherical harmonics ${}_sY_{lm}$ do form an orthonormal set for fixed s. It is easy to show that for $l \le L$ the ${}_sY_{lm}$ are given uniquely by the ${}_sZ^L_{m'm}$:

$$
{}_sY_{lm} = \sum_{m_1=0}^{L-s}\sum_{m_2=0}^{L+s} {}_sB^{L\ m_1m_2}_{lm}\,{}_sZ^L_{m_1m_2}, \tag{4.13}
$$

$$
s \le l \le L, \quad |m| \le l;
$$

$$
{}_sB^{L\ m_1m_2}_{lm} = \frac{a_{lm}}{[(l-s)!\,(l+s)!]^{\frac{1}{2}}}\sum_{p=0}^{p_m}(-1)^{p+s-m}\binom{l-s}{p} \times \binom{l+s}{p+s-m}\binom{L-l}{L-s-m_1-p} \times \delta_{m_2,m_1+s+m}, \tag{4.14}
$$

$$
p_m = \min\{L-s-m_1,\ l-s,\ l+m\} \tag{4.14a}
$$

and the a_{lm} are the constants defined in Eq. (2.11).

For fixed s and L the coefficients ${}_sB^{L\ m_1m_2}_{lm}$ form a nonsingular

$$
(L-s+1)(L+s+1) \times (L-s+1)(L+s+1)
$$

matrix $[(l, m), (m_1, m_2)]$ connecting the ${}_sZ^L_{m_1m_2}$ to the ${}_sY_{lm}$. Since the ${}_sZ^L_{m_1m_2}$ transform under the

$$
\mathfrak{D}^{[\frac{1}{2}(L-s)][\frac{1}{2}(L+s)]}
$$

representation of the Lorentz group up to the factor $K^{-L}e^{il\lambda}$, it follows that the ${}_sY_{lm}(|s| \le l \le L$ and $|m| \le l)$ transform under an equivalent representation up to the same factor.

The above results hold both for L and s integral, or half-integral.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Invariant Transformations and Newman–Penrose Constants*

JOSHUA N. GOLDBERG
Physics Department, Syracuse University, Syracuse, New York

(Received 29 November 1966)

The relationship between constants of the motion and invariant transformations is discussed. Particular emphasis is placed on strong conservation laws which are of the form $\mathcal{U}^{\mu\nu}_{\lambda,\mu\nu} \equiv 0$. The existence of such a law leads to constants of the motion which are surface integrals and therefore generally do not generate an invariant transformation. However, when there is an associated weak conservation law, such that $t^{\mu}{}_{,\mu} \equiv -\delta y_A L^A$ ($L^A = 0$ are the field equations and δy_A the invariant change in field variables), a nontrivial invariant transformation exists. These results are applied to the discussion of the Newman–Penrose constants for the electromagnetic field. The conclusion arrived at is $\delta y_A = 0$, which suggests that the invariant transformation generated by the Newman–Penrose constants is trivial.

1. INTRODUCTION

A NEW set of conserved quantities for free massless fields has been discovered by Newman and Penrose.[1] For linear field theories, an infinite hierarchy of such conserved quantities can be defined. However, the nonlinear field equations of general relativity permit only the first set of these constants to exist and then only in asymptotically flat space–times. It is well known[2] that constants of the motion generate an invariant transformation, and conversely, that the

* This research was supported in part by the Aerospace Research Laboratories.

[1] E. T. Newman and R. Penrose, Phys. Rev. Letters **15**, 231 (1965).

[2] For example, see P. G. Bergmann, *Encyclopedia of Physics*, Vol. **54** (Springer-Verlag, Berlin, S. Flugge, Ed., 1962); A Trautman, "Conservation Laws," article in *Gravitation*, L. Witten, Ed. (John Wiley & Sons, Inc., New York, 1962).

generator of an invariant transformation is conserved. Therefore, it is natural to ask for the invariant transformation generated by these Newman–Penrose (N–P) constants of the motion.

In this paper, a preliminary attempt is made to answer the above question. The N–P constants are defined by integration over a two-dimensional closed surface. Thus, the integral can be defined by a skew tensor (or more generally, a complex) which is a superpotential associated with the transformation. Such a superpotential results from a transformation law which contains arbitrary functions.[2] In Sec. 2 we sketch the relationship between an invariant transformation, the associated conservation law, and the resulting superpotential. The electromagnetic field is the best known example of a linear field theory which exhibits the essential properties to be found in all linear massless free fields. Therefore, Sec. 3 contains the explicit derivation of the N–P constants for the Maxwell field and a discussion of their transformation properties. The superpotentials are constructed in Sec. 4 and a discussion of the suggested invariant transformation is given.

2. INVARIANT TRANSFORMATIONS AND SUPERPOTENTIALS

Whenever a set of field equations can be derived from an action principal,[3]

$$\delta\int \mathfrak{L}(y_A, y_{A,\mu})\, d^4x = 0,$$

the conservation laws associated with an invariant transformation can be derived by Noether's theorems. In the following, we essentially give the discussion ot Trautman.[2]

Let the invariant transformation be given by

$$\bar{\delta} y_A = u^{\nu}_{Aj}\xi^j_{,\nu} + w_{Aj}\xi^j. \tag{2.1}$$

The n quantities $\xi^j(x)$ are the descriptors of the transformation and in general may be a set of arbitrary functions or may simply depend on a finite number of parameters.

For Eq. (1) to define an invariant transformation, the Lagrangian must satisfy

$$\bar{\delta}\mathfrak{L} = Q^{\rho}_{,\rho} = -L^A\bar{\delta} y_A + \left(\frac{\partial\mathfrak{L}}{\partial y_{A,\rho}}\,\bar{\delta} y_A\right)_{,\rho}. \tag{2.2}$$

[3] The $y_A(x)$ represent the N-independent field variables (e.g., A_μ, $g_{\mu\nu}$, ψ_α, etc.); x^μ are the four coordinates ($\mu = 0, 1, 2, 3$); the signature of the metric is taken to be -2. Where it is convenient, partial derivatives with respect to explicit coordinates r and u will be written as ∂_r and ∂_u, respectively.

Define

$$L^A \stackrel{\text{def}}{=} \left(\frac{\partial\mathfrak{L}}{\partial y_{A,\rho}}\right)_{,\rho} - \frac{\partial\mathfrak{L}}{\partial y_A}, \qquad t^\rho \stackrel{\text{def}}{=} Q^\rho - \bar{\delta} y_A \frac{\partial\mathfrak{L}}{\partial y_{A,\rho}}. \tag{2.3}$$

Then from (2.2) we have

$$t^\rho_{,\rho} = -\bar{\delta} y_A L^A \tag{2.4}$$

and whenever the field equations are satisfied, $L^A = 0$, t^ρ satisfies a local conservation law

$$t^\rho_{,\rho} = 0. \tag{2.5}$$

By integrating (2.5) over a suitable four-dimensional domain which extends to spatial infinity, we can define

$$C = \int_\sigma t^\rho\, d\sigma_\rho \tag{2.6}$$

as a constant of the motion if the flux at infinity vanishes. Note that in (2.6), σ is a three-dimensional open spacelike domain.

If the descriptors depend on a finite member of the parameters, there will be one such conserved quality defined for each parameter. However, the situation is markedly different if the ξ^j are arbitrary functions. By the substitution of Eq. (2.1) into the right-hand side of (2.4), we obtain

$$(t^\rho + u^\rho_{Aj}L^A\xi^j)_{,\rho} = [(u^\rho_{Aj}L^A)_{,\rho} - w_{Aj}L^A]\xi^j. \tag{2.7}$$

Integrating (2.7) over an arbitrary four-dimensional domain on whose boundary the descriptors ξ^j vanish, we find that the following identity must be satisfied:

$$(u^\rho_{Aj}L^A)_{,\rho} - w_{Aj}L^A \equiv 0.$$

From this identity it follows that for all ξ^j

$$(t^\rho + u^\rho_{Aj}L^A\xi^j)_{,\rho} \equiv 0.$$

Hence, there exists a skew quantity $\mathfrak{U}^{\rho\sigma}_{\vee}$ such that

$$t^\rho + u^\rho_{Aj}L^A\xi^j \equiv \mathfrak{U}^{\rho\sigma}_{\vee\,,\sigma}. \tag{2.8}$$

The superpotential $\mathfrak{U}^{\rho\sigma}_{\vee}$ is antisymmetric in (ρ,σ), as indicated by the inverted carat, and clearly satisfies

$$\mathfrak{U}^{\rho\sigma}_{\vee\,,\rho\sigma} \equiv 0, \tag{2.9}$$

a strong conservation law which implies the related weak law (2.5). From (2.8) and a further application of Stokes' theorem when $L^A = 0$, the conserved quantity in Eq. (2.6) can be written as a two-dimensional surface integral,

$$C = \int_\sigma t^\rho\, d\sigma_\rho = \tfrac{1}{2}\oint_{\partial\sigma} \mathfrak{U}^{\rho\sigma}_{\vee}\, d\sigma_{\rho\sigma}. \tag{2.10}$$

Thus, there is a clear cut procedure for determining a constant of the motion if one knows the invariant transformation. Conversely, if a constant of the motion is given, one can try to reverse the steps in order to determine the invariant transformation which is generated by that constant. If the constant of the motion is given as a three-dimensional integral, as in (2.6), one would want to construct a vector density (at least with respect to affine transformations) with a vanishing divergence modulo the field equations. The invariant transformation is then to be identified by (2.4). On the other hand, if the given constant of the motion is a two-dimensional surface integral, one must first construct a superpotential as in (2.10). Then relations of the form (2.8) and (2.4) must be sought before the invariant transformation can be identified.

Given a skew complex $\mathcal{U}^{\rho\sigma}_{\nu}$, the strong conservation law (2.9) is certainly satisfied. Generally, but not always, $\mathcal{U}^{\rho\sigma}_{\nu}$ already contains first derivatives of field variables. Thus, $\mathcal{U}^{\rho\sigma}_{\nu,\sigma}$ contains second derivatives. One can use the field equation $L^A = 0$ in an attempt to eliminate the second time derivatives. Such a procedure leads to a new set of quantities:

$$t^\rho \stackrel{\text{def}}{=} \mathcal{U}^{\rho\sigma}_{,\sigma} - \Lambda^\rho_A L^A \tag{2.11}$$

which, of course, satisfies a weak conservation law

$$t^\rho_{,\rho} = 0;$$

that is to say, a conservation law that is satisfied only when $L^A = 0$. In general, however, $t^\rho_{,\rho}$ does not immediately have the form (2.4), whose right-hand side is homogeneous linear in the field equations themselves. That is, the right-hand side generally contains derivatives of the field equations as follows:

$$t^\rho_{,\rho} \equiv \alpha_A L^A + \beta^\rho_A L^A_{,\rho}.$$

Defining

$$t^{*\rho} = t^\rho - \beta^\rho_A L^A, \tag{2.12}$$

one finds

$$t^{*\rho}_{,\rho} = (\alpha_A - \beta^\rho_{A,\rho})L^A$$

which is of the form (2.4). The invariant transformation is then identified as

$$\bar{\delta} y_A = -(\alpha_A - \beta^\rho_{A,\rho}). \tag{2.13}$$

For an arbitrarily chosen complex $\mathcal{U}^{\rho\sigma}_{\nu}$, one expects to find $\bar{\delta} y_A = 0$, for α_A differs from $\beta^\rho_{A,\rho}$ only when a nontrivial invariant transformation exists.[4]

We propose to carry out the above construction in order to determine the invariant transformation, if any, generated by the N–P constants. It is carried out in Sec. 4 after the N–P constants have been identified in the next section.

[4] I want to thank Professor P. Bergmann for an extensive discussion concerning this section.

3. MAXWELL'S EQUATIONS

To simplify our discussion, it is convenient to introduce a null tetrad as in the work of Newman and Penrose.[1] Let the metric have the form

$$ds^2 = du(du + 2\,dr) - r^2\,d\theta^2 - r^2 \sin^2\theta\,d\varphi^2 \tag{3.1}$$

and define a null tetrad (see Fig. 1)

$$\sqrt{2}\,l_\mu = \delta^0_\mu, \quad \sqrt{2}\,n_\mu = \delta^0_\mu + 2\delta'_\mu,$$
$$\sqrt{2}\,m_\mu = -r(\delta^2_\mu + i\sin\theta\,\delta^3_\mu). \tag{3.2}$$

With this choice we have

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1 \tag{3.3}$$

while all other contractions vanish.

The following self-dual (up to a factor $-i$) bivectors are also useful:

$$V^{\mu\nu} = l^\mu m^\nu - l^\nu m^\mu,$$
$$M^{\mu\nu} = l^\mu n^\nu - l^\nu n^\mu - m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu,$$
$$U^{\mu\nu} = n^\mu \bar{m}^\nu - n^\nu \bar{m}^\mu. \tag{3.4}$$

Using these bivectors, the self-dual electric field tensor

$$F^{(-)\mu\nu} = F^{\mu\nu} + i\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

can be expressed in terms of three complex scalars,

$$F^{(-)\mu\nu} = \phi_2 V^{\mu\nu} - \phi_1 M^{\mu\nu} - \phi_0 U^{\mu\nu}, \tag{3.5}$$
$$\phi_0 = -\tfrac{1}{2} V_{\mu\nu} F^{(-)\mu\nu},$$
$$\phi_1 = \tfrac{1}{4} M_{\mu\nu} F^{(-)\mu\nu},$$
$$\phi_2 = \tfrac{1}{2} U_{\mu\nu} F^{(-)\mu\nu}. \tag{3.6}$$

Maxwell's equations

$$F^{(-)\mu\nu}_{;\nu} = 0$$

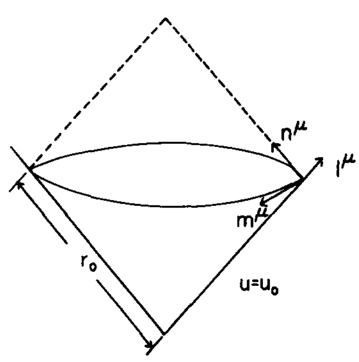


Fig. 1. The null surface $u = u_0$ showing an embedded two surface $r = r_0$ together with the tetrad vectors l^μ, n^μ, and m^μ. The dotted lines indicate the retrograde cone containing the same two surface.

can then be expressed in terms of these qualities as

$$\frac{1}{r^2}\partial_r(r^2\phi_1) - \frac{1}{r}\bar{\eth}\phi_0 = 0, \tag{3.7a}$$

$$\frac{1}{r}\partial_r(r\phi_2) - \frac{1}{r}\bar{\eth}\phi_1 = 0, \tag{3.7b}$$

$$\frac{1}{r}(2\partial_u - \partial_r)(r\phi_0) - \frac{1}{r}\eth\phi_1 = 0, \tag{3.7c}$$

$$\frac{1}{r^2}(2\partial_u - \partial_r)(r^2\phi_1) - \frac{1}{r}\eth\phi_2 = 0. \tag{3.7d}$$

The angular differential operator[5] $\eth$ acts on functions of definite spin-weight s to produce a function of spin weight $s + 1$ while $\bar{\eth}$ acts on functions of spin weight s to produce a function of spin weight $s - 1$. Spin weight is determined by the complex vector $\mathbf{m}$ which is defined only up to a phase factor $e^{i\varphi}$. A function ξ is said to have spin weight s if $m \to e^{i\varphi}m \Rightarrow \xi \to e^{is\varphi}\xi$. Clearly ϕ_0, ϕ_1, and ϕ_2 have spin weight $+1$, 0, -1, respectively.

The class of solutions to (3.7), which is of interest to us, has the asymptotic form

$$\phi_0 = \sum_{n=0}^{N}\frac{\phi_0^n(u, \theta, \varphi)}{r^{3+n}} + 0(r^{-4-N}),$$

$$\phi_1 = \frac{\phi_1^0}{r^2} - \sum_{n=0}^{N}\frac{\bar{\eth}\phi_0^n}{(n+1)r^{3+n}} + 0(r^{-4-N}), \tag{3.8}$$

$$\phi_2 = \frac{\phi_2^0}{r} + \frac{\bar{\eth}\phi_1^0}{r^2} - \sum_{n=0}^{N}\frac{\bar{\eth}\bar{\eth}\phi_0^n}{(n+1)(n+2)r^{3+n}} + 0(r^{-4-N}).$$

Equation (3.7d) imposes a relationship between ϕ_1^0 and ϕ_2^0:

$$2\partial_u\phi_1^0 = \eth\phi_2^0. \tag{3.9}$$

This equation is a consequence of charge conservation.

To obtain the N–P constants, eliminate ϕ_1 from (3.7a) and (3.7c), apply $r^2\eth$ to the former and $\partial_r r^3$ to the latter, and add

$$\mathfrak{L}^0 \stackrel{\text{def}}{=} (2\partial_u - \partial_r)r^2\partial_r(r\phi_0) - \eth\bar{\eth}r\phi_0 = 0. \tag{3.10}$$

This equation is of spin weight $+1$ whereas invariant integrals should have spin weight 0. Multiplying the above by ${}_{-1}Y_{lm}(\theta, \varphi)$ and rearranging terms, we get

$$(2\partial_u - \partial_r)\partial_r r^3\phi_0\,{}_{-1}Y_{lm} + \frac{2}{r}\partial_r r^3\phi_0\,{}_{-1}Y_{lm} - (2 - l(l+1))r\phi_0\,{}_{-1}Y_{lm} - \eth[{}_{-1}Y_{lm}(\bar{\eth}r\phi_0)] + \bar{\eth}[r\phi_0(\eth\,{}_{-1}Y_{lm})] = 0.$$

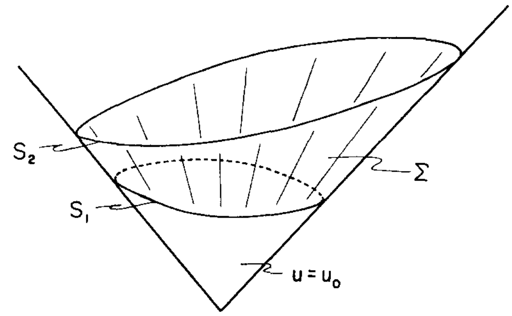


FIG. 2. The curves S_1 and S_2 represent two different spacelike sections of the null cone $u = u_0$. The shaded area Σ is that portion of the cone which is bounded by S_1 and S_2.

The last two terms are a divergence on the sphere, while the third term vanishes for $l = 1$. From the asymptotic behavior of ϕ_0 given in Eq. (3.8), it is easy to show that

$$\partial_u Q_{1m} = 0,$$

$$Q_{1m} = \lim_{r\to\infty}\oint\partial_r(r^3\phi_0)\,{}_{-1}Y_{1m}r^2\sin\theta\,d\theta\,d\varphi,$$
$$m = 0, \pm 1. \tag{3.11}$$

In general one can show

$$\partial_u Q_{lm} = 0, \quad -l \le m \le l,$$

$$Q_{lm} = \lim_{r\to\infty}\int(\partial_r r^2)^l(r\phi_0)\,{}_{-1}Y_{lm}r^2\sin\theta\,d\theta\,d\varphi \tag{3.12}$$

when the limit exists. These are the N–P constants.

4. SURFACE INTEGRALS AND STRONG CONSERVATION LAWS

The N–P constants derived above are all expressed as integrals over a *two*-dimensional spherical surface. From the discussion in Sec. 2, one would expect these constants to be related to a strong conservation law with a superpotential.

We write the superpotential as a linear combination of the bivectors defined in Eq. (3.4),

$$\mathfrak{U}^{\mu\nu} = \Theta M^{\mu\nu} + AV^{\mu\nu} + B\bar{V}^{\mu\nu} + CU^{\mu\nu} + D\bar{U}^{\mu\nu} + E\bar{M}^{\mu\nu}, \tag{4.1}$$

giving the surface integral as[6]

$$\mathfrak{U} = \oint\mathfrak{U}^{\mu\nu}\,d\sigma_{\mu\nu},$$

$$d\sigma_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,d\sigma^{\rho\sigma},$$

$$\epsilon_{0123} = (-g)^{\frac{1}{2}}, \tag{4.2}$$

[5] E. Newman and R. Penrose, J. Math. Phys. 7, 863 (1966); J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan, J. Math. Phys. 8, 2155 (1967). In the Appendix we give a brief summary of the needed properties of functions with spin weight.

[6] In Sec. 2, $\mathfrak{U}^{\nu\mu}$ and t^μ have tensor density character whereas here for convenience, we take them to have zero density weight. The difference is the factor $(-g)^{\frac{1}{2}} = r^2\sin\theta$ in the null coordinates of Eq. (3.1). This accounts for the use of the tensor surface element $d\sigma_{\mu\nu}$ in Eq. (4.2) and for covariant differentiation in the following where ordinary derivatives occur in Sec. 2.

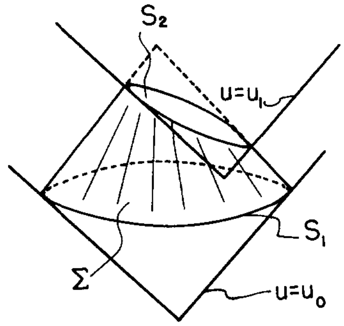


FIG. 3. The curve S_1 represents the surface $(u = u_0,\ r = r_0)$ while the curve S_2 represents the intersection of the null surface $u = u_1$ and the retrograde cone containing S_1. The shaded area is that part of the retrograde cone which is bounded by S_1 and S_2.

where $d\sigma^{\rho\sigma}$ is tensor extension of the two surface. The six coefficients in (4.1) are to be chosen to satisfy the following conditions[7]:

(i) In the limit of null infinity (u = const, $r \to \infty$) $\mathcal{U}$ becomes an N–P const.

(ii) In the limit of null infinity, $\mathcal{U}$ is invariant.

(a) $\lim_{r\to\infty}$ does not depend on the choice of two surface as long as it remains embedded in u = const (see Fig. 2).

(b) $\lim_{r\to\infty}$ does not depend on the choice of null surface u = const (see Fig. 3).

Conditions (ii) show that the N–P constants are in fact constants of the motion.

Choose

$$\Theta = -\partial_r(r^3\phi_0)\,{}_{-1}Y_{1m}, \quad E = 0. \tag{4.3}$$

For the two surface ($u = u_0$, $r = r_0$),

$$d\sigma_{\mu\nu} = l_{[\mu}n_{\nu]}r^2\,d\Omega,$$

and we find from (4.2) and (3.11) that

$$\lim_{r\to\infty} \mathcal{U} = Q_{1m}. \tag{4.4}$$

Thus, the choice (4.3) satisfied condition (i).

(i) In order to satisfy (ii) it is sufficient to show

$$l_\mu \mathcal{U}^{\mu\nu}_{;\nu} = 0\left(\frac{1}{r^4}\right) \tag{4.5a}$$

and

$$n_\mu \mathcal{U}^{\mu\nu}_{;\nu} = 0\left(\frac{1}{r^3}\right), \tag{4.5b}$$

for by Stokes' theorem

$$\oint_{S_2} \mathcal{U}^{\mu\nu}\,d\sigma_{\mu\nu} - \oint_{S_1} \mathcal{U}^{\mu\nu}\,d\sigma_{\mu\nu} = \int_\Sigma \mathcal{U}^{\mu\nu}_{;\nu}\,d\sigma_\mu,$$

where

$$d\sigma_\mu = l_\mu r^2\,dr\,d\Omega$$

on $u = u_0$ and

$$d\sigma_\mu = n_\mu r^2\,dr\,d\Omega$$

on the retrograde null cone. The reason the orders are different in (4.5a) and (4.5b) is that in both cases one takes the limit $r \to \infty$ for $u = u_0$; when both S_1 and S_2 are embedded in $u = u_0$, their separation becomes infinite in the limit, whereas their separation remains constant along each ray if they are related as in Fig. 3 while passing to the limit.

Equation (4.5b) turns out to be the more sensitive condition. With

$$A = r^2\,{}_{-1}Y_{1m}\bar{\eth}\phi_0, \quad B = r^2\phi_0(\eth\,{}_{-1}Y_{1m})$$

and

$$C = D = 0, \tag{4.6}$$

we find [$\mathcal{L}^0$ given by (3.10)]

$$\sqrt{2}\,n_\mu \mathcal{U}^{\mu\nu}_{;\nu} = -\mathcal{L}^0\,{}_{-1}Y_{1m} + \frac{4}{r}\,\partial_r r^3\phi_0\,{}_{-1}Y_{1m}, \tag{4.7a}$$

$$\sqrt{2}\,l_\mu \mathcal{U}^{\mu\nu}_{;\nu} = \frac{1}{r^2}\,\partial_r r^2\partial_r r^3\phi_0\,{}_{-1}Y_{1m}. \tag{4.7b}$$

When $\mathcal{L}^0 = 0$ and ϕ_0 has the asymptotic form given in (3.8), the conditions (4.5 a, b) are satisfied.

From the invariance guaranteed by Eq. (4.5a) together with the transformation properties of[8] ${}_{-1}Y_{1m}$ and the change in ϕ_0 due to the change in the canonical tetrad, one can show that the constants Q_{1m} transform with respect to $D^{(1)(0)}$ representation of the Lorentz group; that is, like the components of a self-dual skew tensor [up to a factor $(-i)$]

A similar discussion can, of course, be carried out for the higher-order N–P constants. Superpotentials can be found for these cases too; the conditions corresponding to (4.7 a, b), however, depend on derivatives of the field equations and the appropriate ${}_{-1}Y_{lm}$. These higher-order constants moreover, do not transform in accordance with an irreducible representation of the Lorentz group by themselves, but rather, add lower-order constants as well.

Now we are ready to ask for the invariant transformation as defined by Eq. (2.4). From Eqs. (4.1), (4.3), (4.6), and (2.8), we find

$$\mathcal{U}^{\rho\sigma} = -\partial_\mu(r^3\phi_0)\,{}_{-1}Y_{1m}M^{\rho\sigma} + r^2\,{}_{-1}Y_{1m}\bar{\eth}\phi_0 V^{\mu\nu} - r^2\phi_0(\eth\,{}_{-1}Y_{1m})\bar{V}^{\mu\nu}, \tag{4.8}$$

$$\mathcal{U}^{\rho\sigma}_{;\sigma} = t^\rho - l^\rho\,{}_{-1}Y_{1m}\mathcal{L}^0,$$

$$t^\rho = l^\rho\left(\frac{2}{r}\,\partial_r r^3\phi_0\,{}_{-1}Y_{1m}\right) - \bar{m}^\rho({}_{-1}Y_{1m}\partial_r r^3\eth\phi_0). \tag{4.9}$$

[7] We assume that the two surface of integration is embedded in a coordinate surface u = const (Fig. 1). This restriction could be dropped without altering the method of the proof or the conclusions to be drawn although the detailed calculations are more difficult. In essence the generalization is treated in Sec. V of the paper by Newman and Penrose (to be published).

[8] See final section of the second reference given in Ref. 5.

From (4.8) we have

$$(r^2 \sin \theta t^\rho)_{,\rho} = (r^2 \sin \theta l^\rho \,{}_{-1}Y_{1m}\mathfrak{L}^0)_{,\rho}.$$

In order to fit the scheme of Eq. (2.4), it is necessary that

$$(r^2 \sin \theta l^\rho \,{}_{-1}Y_{1m}\mathfrak{L}^0)_{,\rho} = -\bar{\delta}\phi_0 \mathfrak{L}^0. \qquad (4.10)$$

Clearly, this is not the case. However, one can add to t^ρ any quantity which vanishes modulo the field equations without altering the conserved quantity; we may define

$$\hat{t}^\rho = t^\rho - l^\rho \,{}_{-1}Y_{1m}\mathfrak{L}^0.$$

Then (4.8) tells us that

$$(r^2 \sin \theta \hat{t}^\rho)_{,\rho} = 0$$

which implies that

$$\bar{\delta}\phi_0 = 0. \qquad (4.11)$$

That is, the invariant transformation, whatever it may be, does not change ϕ_0 at all.

Therefore, in accordance with the discussion at the end of Sec. 3, the generator defined by an N–P constant connected to an invariant transformation and the existence of constants of the motion is fortuitous. That is, the constants of the motion exist only because of the particular choice of boundary conditions. The result is curious, however, because the condition (4.5b) requires $\mathfrak{L}^0 = 0$.

On the other hand, we are familiar with an invariant transformation which also gives (4.11): gauge transformations of the vector potential do not change $F^{\mu\nu}$ and hence ϕ_0. However, an attempt to understand the N–P constants in the framework of the gauge transformation has thus far been unsuccessful. Furthermore, one is not hopeful for such an explanation, for there is no gauge group associated with the scalar field, although the scalar field also exhibits N–P constants. However, it may be an anomalous case.

5. CONCLUSIONS

There is little reason to doubt that Eq. (4.11) correctly expresses the results of the sought for invariant transformation. If we adjoin to $\mathfrak{L}^0$ the second-order equations for ϕ_2 and ϕ_1, an action principle can be constructed as follows:

$$0 = -\delta \int \{\delta_r(r\phi_2)(2\partial_u - \partial_r)(r\phi_0) - \eth(r\phi_2)\bar{\eth}(r\phi_0) + (2\partial_u - \partial_r)(r^2\phi_1)\partial_r(r^2\phi_1) - (\bar{\eth}\phi_1)(\eth\phi_1)\} \sin\theta \, d\theta \, d\phi \, dr \, du.$$

Thus Noether's theorem must hold if an invariant transformation exists. Replace $\bar{\delta}\phi_0$ in Eq. (4.10) by $\bar{\delta}\phi_2$ and add to (4.11) $\bar{\delta}\phi_2 = \bar{\delta}\phi_1 = 0$. In the previous argument leading up to (4.11), we have merely turned the crank in reverse. Given the constant of the motion, find the invariant transformation.

However, the discussion at the end of the previous section does not seem to offer much hope of understanding these constants. Further efforts in coming to an understanding are worthwhile because Einstein's equations exhibit five such complex constants in asymptotically flat spaces, and their existence seems to inhibit the physical behavior of mass distributions.[9]

ACKNOWLEDGMENTS

Finally, the author wants to thank both Dr. Newman and Dr. Penrose for several very interesting discussions about these peculiar constants of the motion.

APPENDIX

The properties of $\eth$ and the spin-s spherical functions are discussed at length in the two papers noted in Ref. 5. Here we give only those properties which we use explicitly.

If ξ is a function of spin-weight s, then

$$\eta = \eth\xi = -(\sin\theta)^s(\partial_\theta + i \csc\theta\partial\varphi)(\sin\theta)^{-s}\xi$$

has spin weights $s + 1$ and

$$\zeta = \bar{\eth}\xi = -(\sin\theta)^{-s}(\partial_\theta - i \csc\theta\partial\varphi)(\sin\theta)^{s}\xi$$

has spin weight $s - 1$. In particular the spin-s spherical harmonics satisfy the following relations:

$$\eth \,{}_sY_{lm}(\theta, \varphi) = [(l - s)(l + s + 1)]^{\frac{1}{2}} \,{}_{s+1}Y_{lm}(\theta, \varphi),$$
$$\bar{\eth} \,{}_sY_{lm}(\theta, \varphi) = -[(l + s)(l - s + 1)]^{\frac{1}{2}} \,{}_{s-1}Y_{lm}(\theta, \varphi),$$
$${}_sY_{lm}(\theta, \varphi) = 0 \quad \text{for} \quad |s| > l,$$
$${}_s\bar{Y}_{lm}(\theta, \varphi) = (-1)^{m+s} \,{}_{-s}Y_{l,-m}(\theta, \varphi),$$

and ${}_0Y_{lm}(\theta, \varphi) = Y_{lm}(\theta, \varphi)$, the usual spherical harmonics. For fixed s the spin-s spherical functions satisfy a completeness relation and orthogonality conditions:

$$\int {}_s\bar{Y}_{lm} \,{}_sY_{l',m'} \sin\theta \, d\theta \, d\varphi = \delta_{ll'}\delta_{mm'}.$$

One further important property is that if ξ has spin weight $+1$, then $\bar{\eth}\xi$ is a divergence on the sphere; similarly, if ξ has spin weight -1, $\eth\xi$ is a divergence on the sphere.

[9] E. T. Newman and R. Penrose, Report to Conference on Relativistic Theories of Gravitation, London (July, 1965).

Solution of the Differential Equation

$$\left(\frac{\partial^2}{\partial x \partial y} + ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cxy + \frac{\partial}{\partial t}\right)P = 0$$

P. LAMBROPOULOS*

Bendix Corporation, Research Laboratories Division, Southfield, Michigan

(Received 8 December 1966)

A series solution of the above differential equation is presented.

1. INTRODUCTION

THE purpose of this paper is to present the solution of a certain partial differential equation in three variables. It appears that the solution of the equation in question is not available in the literature nor does it seem feasible to solve it by conventional methods. The solution presented here is in the form of a series of powers of two of the variables, the coefficients depending on the third.

There is one instance of a physical problem in which a special form of this equation arises. If one considers the density matrix of a harmonic oscillator coupled linearly[1] to a *two*-level system, then in the representation that diagonalizes the Hamiltonian of the oscillator and in lowest-order perturbation theory, one obtains a difference-differential equation for the matrix elements of the density matrix.[2] If Glauber's[3] R function is used as a generating function for the matrix elements, the resulting equation for R is similar to the one discussed here. Since it is possible that the solution of the equation may be useful in a different physical context, it is felt that it might be worth having in the record.

2. METHOD OF SOLUTION

Consider the partial differential equation

$$\left(\frac{\partial^2}{\partial x \partial y} + ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cxy + \frac{\partial}{\partial t}\right)P(x, y, t) = 0, \tag{1}$$

where a, b, c are constants and with the initial condition

$$P(x, y, t = 0) = \Phi(x, y), \tag{2}$$

where $\Phi(x, y)$ is a known function. It is assumed that $\Phi(x, y)$ possesses a Taylor series expansion around the point $(x = 0, y = 0)$.

Let us assume that $P(x, y, t)$ can be written as

$$P(x, y, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{mn}(t)\frac{x^m y^n}{(m!\,n!)^{\frac{1}{2}}} \tag{3}$$

on the condition that such a representation exists. From the initial condition (2) and the assumptions about Φ, we have

$$p_{mn}(0) = \frac{(\partial^{m+n}\Phi/\partial x^m \partial y^n)_{\substack{x=0\\y=0}}}{(m!\,n!)^{\frac{1}{2}}}. \tag{4}$$

Substituting Eq. (3) into Eq. (1) and equating coefficients of equal powers, one obtains

$$-\frac{d}{dt}p_{mn}(t) = amp_{mn}(t) + bnp_{mn}(t) + [(m+1)(n+1)]^{\frac{1}{2}}p_{(m+1)(n+1)}(t) + c(mn)^{\frac{1}{2}}p_{(m-1)(n-1)}(t). \tag{5}$$

Now, for $m \geq n$ we introduce

$$f_{mn} \equiv p_{mn}\left(\frac{m!}{n!}\right)^{\frac{1}{2}} \tag{6a}$$

and for $m \leq n$ we introduce

$$g_{mn} \equiv p_{mn}\left(\frac{n!}{m!}\right)^{\frac{1}{2}}. \tag{6b}$$

Clearly, for $m = n$, $f_{mm} = g_{mm} = p_{mm}$. Upon substitution into Eq. (5), one obtains

$$-\frac{d}{dt}f_{mn} = amf_{mn} + bnf_{mn} + (n+1)f_{(m+1)(n+1)} + cmf_{(m-1)(n-1)} \tag{7a}$$

and

$$-\frac{d}{dt}g_{mn} = amg_{mn} + bng_{mn} + (m+1)g_{(m+1)(n+1)} + cng_{(m-1)(n-1)}. \tag{7b}$$

We now observe that if we consider a subset of f's or g's for which $m - n$ has a fixed value, then the elements of this subset are coupled to each other and

* Present address: Argonne National Laboratory, Argonne, Illinois.

[1] By "linearly" it is meant that the coupling is linear in the creation and annihilation operators of the harmonic oscillator and that the interaction does not change the populations of the *two*-level system appreciably.

[2] G. Hok, M. Barasch, P. Lambropoulos, and E. K. Miller, Air Force Avionics Laboratory, Wright-Patterson Air Force Base, Ohio, Technical Report AFAL-TR-66-27 (1966).

[3] R. Glauber, Phys. Rev. **131**, 2766 (1963).

to no elements of another subset corresponding to a different value of $m-n$. Thus, let us set $m=n+l$ in Eq. (7a) and $n=m+l$ in Eq. (7b). Moreover, to compress notation, we set $f_{(n+l)n}=f_n^{(l)}$ and $g_{m(m+l)}=g_m^{(l)}$. By letting l vary from 0 to ∞, we obtain all coefficients. The equations that we now have are

$$-\frac{d}{dt}f_n^{(l)}=(\beta n+al)f_n^{(l)}+(n+1)f_{n+1}^{(l)}+c(n+l)f_{n-1}^{(l)} \quad (8a)$$

and

$$-\frac{d}{dt}g_m^{(l)}=(\beta m+bl)g_m^{(l)}+(m+1)g_{m+1}^{(l)}+c(m+l)g_{m-1}^{(l)}, \quad (8b)$$

where

$$\beta\equiv a+b. \quad (9)$$

For Eqs. (8) to be consistent with Eqs. (7), $f_{n-1}^{(l)}$ and $g_{n-1}^{(l)}$ are so defined that they vanish for $n=0$.

To solve Eq. (8a), we introduce the generating function $F^{(l)}(Z,t)$ defined by

$$F^{(l)}(Z,t)=\sum_{n=0}^{\infty}f_n^{(l)}(t)Z^n. \quad (10)$$

Multiplying Eq. (8a) by Z^n and summing from 0 to ∞ one finds that $F^{(l)}$ satisfies the following partial differential equation:

$$\frac{\partial F^{(l)}}{\partial t}+(cZ^2+\beta Z+1)\frac{\partial F^{(l)}}{\partial Z}+[c(l+1)Z+al]F^{(l)}=0. \quad (11)$$

Similarly, introducing the generating function

$$G^{(l)}(Z,t)=\sum_{n=0}^{\infty}g_n^{(l)}(t)Z^n, \quad (12)$$

we find that it must satisfy the partial differential equation

$$\frac{\partial G^{(l)}}{\partial t}+(cZ^2+\beta Z+1)\frac{\partial G^{(l)}}{\partial Z}+(c(l+1)Z+bl)G^{(l)}=0. \quad (13)$$

The problem has now been reduced to solving a first-order partial differential equation which can be easily done by the method of characteristics. The program is to solve Eqs. (11) and (13), express the solutions as series of powers of Z and thus determine $f_n^{(l)}(t)$ and $g_n^{(l)}$ from which one can obtain $p_{mn}(t)$ and hence, $P(x,y,t)$.

According to the method of characteristics,[4] Eq. (11) is equivalent to

$$dt=\frac{dZ}{cZ^2+\beta Z+1}=\frac{dF^{(l)}}{(c(l+1)Z+al)F^{(l)}}. \quad (14)$$

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1963), Vol. II.

In the general case in which β, $c\neq 0$, we have

$$cZ^2+\beta Z+1=c(Z-r_1)(Z-r_2), \quad (15)$$

where

$$r_{1,2}=\frac{1}{2}\left\{-\frac{\beta}{c}\pm\left[\left(\frac{\beta}{c}\right)^2-4\right]^{\frac{1}{2}}\right\}. \quad (16)$$

It is now straightforward to solve the first of Eqs. (14). The result is

$$Z(t)=\tfrac{1}{2}(r_1+r_2)+\tfrac{1}{2}(r_2-r_1)\frac{e^{c(r_1-r_2)t}+\nu}{e^{c(r_1-r_2)t}-\nu}, \quad (17)$$

where

$$\nu\equiv\frac{Z_0-r_2}{Z_0-r_1} \quad (18)$$

and $Z_0=Z\ (t=0)$.

Having $Z(t)$, one can solve the second of Eqs. (14), thus obtaining

$$F^{(l)}(Z,t)=F_0^{(l)}(Z_0)e^{(cr_1(l+1)+al)t}\left(\frac{1-\nu}{e^{c(r_1-r_2)t}-\nu}\right)^{l+1}, \quad (19)$$

where

$$F_0^{(l)}(Z_0)\equiv\sum_{n=0}^{\infty}f_n^{(l)}(0)Z_0^n. \quad (20)$$

From Eqs. (17) and (18), one obtains

$$\nu=\exp\left[c(r_1-r_2)t\right]\frac{Z-r_2}{Z-r_1} \quad (21)$$

and

$$Z_0=\tfrac{1}{2}(r_1+r_2)+\tfrac{1}{2}(r_1-r_2)\times\frac{\exp\left[c(r_1-r_2)t\right]+(Z-r_1)/(Z-r_2)}{\exp\left[c(r_1-r_2)t\right]-(Z-r_1)/(Z-r_2)}. \quad (22)$$

Substituting into Eq. (19), one has an expression for $F_n^{(l)}(Z,t)$ from which one can calculate

$$f_n^{(l)}(t)=\frac{1}{n!}\left[\frac{\partial^n}{\partial Z^n}F^{(l)}(Z,t)\right]_{Z=0}. \quad (23)$$

To obtain $g_n^{(l)}(t)$, one simply replaces a by b and $f_n^{(l)}(0)$ by $g_n^{(l)}(0)$ in the expression for $f_n^{(l)}(t)$ as one can see by simply comparing Eq. (13) to Eq. (11).

This completes the formal solution of the problem. The resulting series for $P(x,y,t)$ is fairly cumbersome and we refrain from presenting it here, since it only involves some further algebraic manipulations. We do, however, present the final result for two special cases which are relatively simpler and potentially more useful.

A. Special Case I: $c=0$, $\beta\neq 0$

To handle this special case, one has to go back to Eq. (14), because Eq. (17) is not valid for $c=0$. Then we obtain

$$dt=\frac{dZ}{\beta Z+1}=\frac{dF^{(l)}}{alF^{(l)}}. \quad (24)$$

These equations give

$$Z(t) = (Z_0 + \beta^{-1})e^{\beta t} - \beta^{-1} \tag{25}$$

and

$$F^{(l)}(Z, t) = F_0^{(l)}(Z_0)e^{alt} = e^{alt}\sum_{n=0}^{\infty} f_n^{(l)}(0)Z_0^n. \tag{26}$$

Using Eq. (25), Z_0 can be expressed in terms of Z and t. Substituting into Eq. (26) we have

$$F^{(l)}(Z, t) = e^{alt}\sum_{n=0}^{\infty} f_n^{(l)}(0)[(Z + \beta^{-1})e^{-\beta t} - \beta^{-1}]^n. \tag{27}$$

Application of the binomial theorem twice leads to

$$F^{(l)}(Z, t) = e^{alt}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{q=0}^{n-r} f_n^{(l)}(0)(-1)^r \times \frac{n!}{r!\,q!\,(n-r-q)!}\, e^{-(n-r)\beta t}\beta^{-(q+r)}Z^{n-r-q}, \tag{28}$$

which upon interchanging summations can be written as

$$F^{(l)}(Z, t) = \sum_{s=0}^{\infty} Z^s e^{alt}\sum_{n=s}^{\infty}\sum_{r=0}^{n-s} f_n^{(l)}(0)(-1)^r \times \frac{n!}{r!\,s!\,(n-r-s)!}\, e^{-(n-r)\beta t}\beta^{s-n}. \tag{29}$$

From this equation, we obtain

$$f_m^{(l)}(t) = e^{alt}\sum_{n=m}^{\infty}\sum_{r=0}^{n-m} f_n^{(l)}(0)(-1)^r \times \frac{n!}{r!\,m!\,(n-r-m)!}\, e^{-(n-r)\beta t}\beta^{m-n}. \tag{30}$$

To obtain $g_m^{(l)}(t)$, we replace $f_n^{(l)}(0)$ by $g_n^{(l)}(0)$ and multiply the above equation by $\exp(b-a)lt$; to obtain $f_m^{(0)}(t) = p_{mm}(t)$, we simply set $l = 0$. Now using Eqs. (6), we can determine $p_{mn}(t)$ for all m,n and the final solution for $P(x, y, t)$ in this special case is

$$\begin{aligned} P_I(x, y, t) &= \sum_{m=0}^{\infty}\frac{x^m y^m}{m!}\sum_{n=m}^{\infty}\sum_{r=0}^{n-m} p_n(0)(-1)^r \\ &\times \frac{n!}{r!\,m!\,(n-r-m)!}\, e^{-(n-r)\beta t}\beta^{m-n} \\ &+ \sum_{l=1}^{\infty}\sum_{m=0}^{\infty} 1/(m+l)!\sum_{n=m}^{\infty}\sum_{r=0}^{n-m} \\ &\times \{e^{alt}p_{(n+l)n}(0)x^{m+l}y^m + e^{blt}p_{n(n+l)}(0)x^m y^{m+l}\}(-1)^r \\ &\times \frac{n!}{r!\,m!\,(n-r-m)!}\, e^{-(n-r)\beta t}\beta^{m-n}. \end{aligned} \tag{31}$$

To verify that the initial condition is satisfied, one should note that

$$\sum_{r=0}^{m}\frac{(-1)^r}{r!\,(m-r)!} = \delta_{m0}, \tag{32}$$

where δ_{m0} is the Kronecker delta.

B. Special Case II: $a = b = c = 0$

For this case, Eq. (14) reduces to

$$\frac{\partial F^{(l)}}{\partial t} = \frac{\partial F^{(l)}}{\partial Z} \tag{33}$$

which is satisfied by any function of the form $F^{(l)}(Z + t)$. Clearly, in order to satisfy the initial condition we must take

$$F^{(l)}(Z, t) = \sum_{n=0}^{\infty} f_n^{(l)}(0)(Z + t)^n. \tag{34}$$

Using the binomial theorem, interchanging summations, etc., we obtain

$$f_n^{(l)}(t) = \sum_{r=n}^{\infty} f_r^{(l)}(0)\frac{r!}{n!\,(r-n)!}\, t^{r-n}. \tag{35}$$

Note that in this special case, $g_n^{(l)}(t)$ is obtained by simply replacing $f_n^{(l)}(0)$ by $g_n^{(l)}(0)$ in Eq. (35). Again using Eqs. (6) to express f and g in terms of the p's and after some rearrangement one obtains

$$P_{II}(x, y, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{\partial^{m+n+2q}\Phi}{\partial x^{m+q}\partial y^{n+q}}\right)_{\substack{x=0\\y=0}} \times \frac{[(m+q)!\,(n+q)!]^{\frac{1}{2}}}{m!\,n!\,q!}\, x^m y^n t^q. \tag{36}$$

It should be noted that in this special case, one could separate variables and solve the resulting partial differential equation in two variables by using a *two*-dimensional Fourier transform. This, however, is possible only when $\Phi(x, y)$ can be expressed as a *two*-dimensional Fourier integral.

ACKNOWLEDGMENT

The author wishes to thank Dr. E. Marom for a discussion.

New Form of Characteristic Functional for Cascade Processes

W. T. SCOTT
Department of Physics, University of Nevada, Reno, Nevada

(Received 20 December 1966)

The characteristic functional for an energy-dependent cascade process is redefined so as to introduce the arbitrary function without using exponentials. An immediate consequence is that the functional becomes the function at $t = 0$ when the process is multiplicative with one particle at the initial time. A simple one-component model of a cosmic-ray shower is used to illustrate how this definition leads directly to (a) a simple form of the Chapman–Kolmogoroff equation, (b) forward and backward integro-differential equations for the functional, and (c) the derivation of all probability functions of interest for the process. An example is given of the distributions in number of particles at different depths when the total cross section for multiplication is proportional to the energy. Some general forms of functional are proposed. The extension to electron–photon showers is outlined.

I. THE CHARACTERISTIC FUNCTIONAL

CHARACTERISTIC functionals have found considerable use in treating stochastic processes. It appears, however, that with a change of definition such functionals can be even more useful. It is the purpose of this article to present a modified definition[1] of the characteristic functional and illustrate some of its consequences.

An example of an energy-conserving cascade shower with energy loss will be used, but the method is not restricted to such processes. We assume that we have particles that can duplicate themselves, i.e., a single one converts into a pair of like particles. The probability per thickness dt of matter traversed that a duplication occurs will be taken as a given function $q(E, u)\, du\, dt$ of the energy E of the original particle and the energy u of either of the daughters, the other having energy $E - u$. We assume symmetry between the daughters:

$$q(E, u) = q(E, E - u). \tag{1}$$

The total probability of a duplication occurring in thickness dt is therefore

$$B(E)\, dt = dt \int_0^E q(E, u)\, du. \tag{2}$$

We further assume that in a path length dt, any particle loses energy $\beta(E)\, dt$. No scattering leading to angular or radial dispersion is considered. Finally, we assume that once a particle reaches energy zero it is lost (or ceases to be counted).

Suppose now we have one particle of energy E_0 incident normally on a layer of matter at $t = 0$. A complete description of the statistics of the shower at any thickness t is contained in the set of master functions[2,3]

$$P_N(E_0; E_1, E_2, \cdots, E_N; t)\, dE_1, \cdots, dE_N, \tag{3}$$

which give the probability that at t there are exactly N particles of energy E_1 to $E_1 + dE_1$, E_2 to $E_2 + dE_2, \cdots, E_N$ to $E_N + dE_N$. The initial condition on P_N is clearly

$$P_N(E_0; E_1, \cdots, E_N; 0) = \delta_{N1}\delta(E_1 - E_0). \tag{4}$$

We define the characteristic functional of the process by the equation

$$C\{\sigma(E); E_0, t\} = \sum_{N=0}^{\infty} \frac{1}{N} \int_0^\infty dE_1\sigma(E_1) \times \int_0^\infty dE_2\sigma(E_2) \cdots \int_0^\infty dE_N\sigma(E_N) P_N(E_0; E_1, \cdots, E_N, t). \tag{5}$$

The variable C is a functional of the arbitrary function $\sigma(E)$ and an ordinary function of the parameters E_0 and t. The letter E on the left-hand side of (5) is of course a dummy variable and will only be included when needed for clarity. An immediate consequence of this definition over the usual one[4] in which $\exp i\theta(E)$ is used in place of $\sigma(E)$ is that C becomes $\sigma(E_0)$ at $t = 0$:

$$C\{\sigma(E); E_0, 0\} = \sigma(E_0), \tag{6}$$

as is easily seen from (4) and (5). The variable C as a function of E_0 and its arbitrary function-argument belong to the same set of functions and become identical at $t = 0$.

The normalization property of P_N becomes merely that $C = 1$ when σ is the constant 1:

$$C\{1; E_0, t\} = 1. \tag{7}$$

Consequently, convergence of the integrals and of the sum in (5) are assured if $\sigma(E)$ is a positive-valued integrable function less than or equal to unity. The characteristic functional will then have the same property:

$$0 \le \sigma(E) \le L; \quad 0 \le C\{\sigma; E_0, t\} \le 1. \tag{8}$$

[1] Earlier given in W. T. Scott, Phys. Rev. **86**, A590 (1952).
[2] W. T. Scott, Phys. Rev. **82**, 893 (1951).
[3] H. J. Bhabha, Proc. Roy. Soc. (London) **A202**, 301 (1950).
[4] See, for instance, M. S. Bartlett, *An Introduction to Stochastic Processes* (Cambridge University Press, Cambridge, England, 1955), pp. 80 and 148.

A restriction on the function $\sigma(E)$ is that it be unity when $E = 0$. In fact, if $E_0 \to 0$, the shower disappears. We assume that $\beta(E)$ remains finite as $E \to 0$, so that a particle of nearly zero energy becomes lost to the shower almost immediately. Consequently, each $P_N \to 0$ for $t > 0$, except that $P_0(E_0, t) \to 1$. As a result we have

$$C\{\sigma(E); 0, t\} = 1, \tag{9}$$

which entails from (6) that

$$\sigma(0) = 1. \tag{10}$$

Note that $\sigma(E)$ is not required to be continuous, although it must be integrable, so the condition (10) needs only to hold *at* $E = 0$. The restriction (10) will not hold if $\beta(E) \equiv 0$, for then the shower does not die out.

II. DERIVED FUNCTIONS

Various generating functions may be found from C. In the first place, let $\sigma(E)$ be a constant $z < 1$ (except at $E = 0$). Then C becomes the generating function for $P(N, E_0, t)$, the probability that there be N particles of any energy at depth t when the initiating energy is E_0:

$$C\{z, E_0, t\} = G(z, E_0, t) = \sum_{N=0}^{\infty} z^N P(N, E_0, t). \tag{11}$$

(Note that when each E_i is integrated from 0 to ∞, all permutations of N particles are counted.)

Now let $\eta(E - E')$ be the step function that is 1 when $E > E'$ and 0 when $E < E'$. Then

$$C\{1 + (z-1)\eta(E - E'); E_0, t\} = g(z, E_0, E', t) = \sum_{N=0}^{\infty} z^N p(N, E_0, E', t), \tag{12}$$

the generating function for the probability $p(N, E', E_0, t)$ of finding N particles of energy greater than E', regardless of the number with lesser energy.

The functional derivative of C with respect to σ, calculated at an energy E_1, is defined in the usual way by adding an infinitesimal delta-function increment $\epsilon\delta(E - E_1)$ to $\sigma(E)$ at $E = E_1$ and taking the limit of the increment of C to the infinitesimal multiplier ϵ. The derivative will be denoted by a subscript 1 and by the inclusion of E_1 as an argument:

$$\begin{aligned} C_1\{\sigma(E); E_0; E_1; t\} &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} [C\{\sigma + \epsilon\delta(E - E_1); E_0, t\} - C\{\sigma; E_0, t\}] \\ &= (d/d\epsilon) C\{\sigma + \epsilon\delta(E - E_1); E_0, t\}\big|_{\epsilon=0}. \end{aligned} \tag{13}$$

Functional differentiation may be carried out on the series in (5) by simply replacing each σ by $\delta(E - E_1)$, one at a time. Repeated derivatives are defined in the same way, and denoted by successive subscripts 2, 3, 4, $\cdots$ and the inclusion of more new arguments $E_2, E_3, E_4, \cdots$.

Now it is easy to see how to find the P's from C: We have

$$\begin{aligned} C\{0; E_0, t\} &= P_0(E_0, t), \\ C_1\{0; E_0; E_1; t\} &= P_1(E_0; E_1; t), \\ C_M\{0; E_0; E_1, \cdots, E_M; t\} &= P_M(E_0; E_1, \cdots, E_M; t). \end{aligned} \tag{14}$$

In fact, Eq. (5) is just the functional equivalent of Maclaurin's expansion.[5]

Consider now what happens if we set $\sigma = 1$, in the Mth functional derivative. Each term in (5) from the Mth onward will have lost M σ's and had the corresponding integrations replaced with fixed energy values. Each distinct physical case will be counted $N!/(N-M)!$ times in the Nth term. We find

$$\begin{aligned} C_M\{1; E_0; E_1; \cdots E_M; t\} &= \sum_{N=M}^{\infty} \frac{1}{(N-M)!} \int_0^{\infty} dE_{M+1} \cdots \int_0^{\infty} dE_N \\ &\quad \times P_N(E_0; E_1, E_2 \cdots E_M, E_{M+1}, \cdots, E_N; t) \\ &= K_M(E_0; E_1, E_2, \cdots, E_M; t), \end{aligned} \tag{15}$$

where $K_M \, dE_1 \, dE_2 \cdots dE_M$ is seen to represent the probability that there be (at t) M particles in the energy ranges $dE_1, dE_2, \cdots, dE_M$ around $E_1, E_2, \cdots, E_M$, and *any number of other particles*.

The function $K_1(E_0; E_1; t)$ is of special importance. The probability of finding a particle in E_1 to $E_1 + dE_1$ regardless of what other particles may be present is also the mean number in that interval, so this function represents the mean-energy spectrum in the shower at depth t. The integral of K_1 over E_1 therefore represents the mean number of particles:

$$\langle N \rangle_{\text{av}} = \sum_{N=0}^{\infty} N P(N, E_0, t) = \int_0^{\infty} dE_1 K_1(E_0, E_1, t). \tag{16}$$

It may be shown by ordinary probabilistic reasoning that the integrals of the K_M give the combinatoric moments

$$\begin{aligned} K(M, E_0, t) &= \langle N(N-1) \cdots (N-M+1) \rangle_{\text{av}} \\ &= \int_0^{\infty} dE_1 \cdots dE_M K_M(E_0; E_1, E_2, \cdots, E_M, t). \end{aligned} \tag{17}$$

The K_M are clearly the coefficients for a generalized Taylor's expansion of $C\{\sigma; E_0, t\}$ around the function

[5] V. Volterra, *Theory of Functionals* (Dover Publications, Inc., New York, 1962), p. 26.

$\sigma(E) = 1$. Thus we can write

$$C\{\sigma; E_0, t\} = \sum_{M=0}^{\infty} \frac{1}{M!} \int_0^{\infty} dE_1[\sigma(E_1) - 1] \times \int_0^{\infty} dE_2[\sigma(E_2) - 1] \cdots \int_0^{\infty} dE_M[\sigma(E_M) - 1] \times K_M(E_0; E_1, E_2, \cdots, E_M, t). \quad (18)$$

If we put $\sigma(E) = z$ in (18), expand the binomial, and compare the results with (11), we have the combinatory result

$$P(N, E_0, t) = \sum_{M=N}^{\infty} \frac{(-1)^{M-N} K(M, E_0, t)}{N!(M-N)!}. \quad (19)$$

It was shown in an earlier work[2] that $K_1(E_0; E_1; t)$ is a Green's function from which all the higher K_M may be calculated in order. Hence C can be found in principle from K_1, but $P(N, E_0, t)$ can be found more directly via K in (19). As usual, we can find $K(M, E_0, t)$ by expanding $G(z, E_0, t)$ in (11) in powers of $z - 1$, since it follows from (11) that

$$G(z, E_0, t) = \sum_{M=0}^{\infty} K(M, E_0, t) \frac{(z-1)^M}{M!}. \quad (20)$$

III. CHAPMAN–KOLMOGOROFF EQUATION

The various forms of the Chapman–Kolmogoroff equation[6] are expressions of the Markoff character of stochastic processes, namely the fact that the intermediate state of the process at any time t is the initial condition for the process following t. The equation takes a particularly simple form when written in terms of C. We have in fact that the function argument σ is simply C at $t = 0$. Generalizing, we have

$$C\{\sigma; E_0, t + t'\} = C\{C\{\sigma, E, t'\}; E_0, t\} \quad (21)$$

for any two times t and t'. A formal proof of (21) may be constructed using standard probabilistic arguments, taking the energies $E_1 \cdots E_N$ for each possible shower at depth t' as the initial values for N independent showers starting at t' and observed at $t + t'$. The argument is lengthy but not difficult and is omitted here.

An explicit formula for $C\{\sigma; E_0, t\}$ may be described as a rule for transforming a function $\sigma(E_0)$ associated with $t = 0$ to another function $C(E_0)$ associated with time t. As t varies, C follows a trajectory in the function space of the functions σ. Since the P_N are uniquely determined by the initial energy and the functions $q(E, u)$ and $\beta(E)$, these trajectories must be nonintersecting.

For any fixed E_0 and any $\beta(E) > 0$, a shower has only a finite extent limited by the penetration depth of the origin particle in the (rare) event that it undergoes no duplication at all. We have

$$t_{\max} = \int_0^{E_0} \frac{dE}{\beta(E)}. \quad (22)$$

Thus we must have

$$C\{\sigma; E_0; t_{\max}(E_0)\} = P_0(E_0, t_{\max}) = 1, \quad (23)$$

showing that C approaches unity in successively more and more dimensions of the function space (each representing a value of E_0) as t increases indefinitely. Equation (23) is seen to be an extension of the result given in (19).

The Chapman–Kolmogoroff relation (21) may be used to define C for negative times by setting $t + t' = 0$. However, this definition will only hold for σ's and values of t that do not violate (9). As t gets more and more negative, any σ except $\sigma \equiv 1$ will sooner or later violate this restriction.

IV. INFINITESIMAL TRANSFORMATION GENERATOR

The variation of C with t is determined by the physical processes expressed by $q(E, u)$ and $\beta(E)$. We can find the generator of an infinitesimal transformation of C from t to $t + dt$, or equivalently from 0 to dt, by considering only the physical processes of zero and first order in dt. We have in fact

$$P_1(E_0; E_1; dt) \simeq [1 - B(E_0)\, dt]\delta[E_1 - E_0 + \beta(E_0)\, dt],$$
$$P_2(E_0; E_1, E_2; dt) \simeq [q(E_0, E_1) + q(E_0, E_2)]\, dt\delta(E_0 - E_1 - E_2),$$
$$P_n \simeq 0; N = 0, 3, 4, 5, \cdots, \quad (24)$$

from which follows

$$C\{\sigma; E_0; dt\} \simeq \sigma(E_0) + dtH\{\sigma; E_0\}, \quad (25)$$

where the functional H is the infinitesimal generator, given by

$$H\{\sigma; E\} = -B(E)\sigma(E) - \beta(E)\, d\sigma(E)/dE + \int_0^E du q(E, \mu)\sigma(u)\sigma(E-u). \quad (26)$$

Using (21), we find immediately that

$$C\{\sigma; E_0; t + dt\} = C\{\sigma; E_0, t\} + dt\, H\{C\{\sigma; E, t\}; E_0\} \quad (27)$$

from which we find the nonlinear integro-differential

[6] W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, Inc., New York, 1957), Vol. I, 2nd ed., pp. 370 and 424.

equation

$$\partial C\{\sigma; E_0, t\}/\partial t = H\{C\{\sigma; E; t\}; E_0\}. \quad (28)$$

Another equation for $\partial C/\partial t$ may be obtained if we first consider a general property of the functional derivative. It follows from the linearity of first-order variations that if $\tau(E)$ is an arbitrary function and ϵ a sufficiently small positive number,

$$C\{\sigma(E) + \epsilon\tau(E); E_0, t\} \simeq C\{\sigma(E); E_0, t) + \epsilon\int_0^\infty du\tau(u)C_1\{\sigma; E_0; u; t\} \quad (29)$$

to first order in ϵ.

If we now replace t' in (20) by dt, we find

$$\frac{\partial C\{\sigma; E_0, t\}}{\partial t} = \int_0^\infty duC_1\{\sigma; E_0; u; t\}H\{\sigma; u\}. \quad (30)$$

Equation (30) is linear in C and C_1, but since both the initial condition and the physical functions q and β are contained in H, there appears to be no way to exploit this linearity by superposing different C's. This equation relates to processes near $t = 0$ and may be called the "forward" equation, to fit the usual terminology,[7] whereas Eq. (28) may be called the "backward" equation. In a certain sense Eqs. (28) and (30) are adjoint to each other.[2] They also represent two special cases of the equation obtained by replacing t' by $t' + dt$ in (21):

$$\frac{\partial C\{\sigma; E_0, t + t'\}}{\partial t} = \int_0^\infty duC_1\{C\{\sigma; E; t'\}; E_0; u; t\} \times H\{C\{\sigma; E, t'\}; u\}. \quad (31)$$

If $t' \to 0$, we obtain (30), whereas if $t \to 0$, we find (28), since we see, for instance from (28), that

$$C_1\{\sigma; E_0; E_1; t\} = \delta(E_1 - E_0). \quad (32)$$

By functionally differentiating (28) and (30), we can find two equations for $C_1\{\sigma; E_0; E_1; t\}$:

$$\frac{\partial C_1\{\sigma; E_0; E_1; t\}}{\partial t} = \int_0^\infty duH_1\{C\{\sigma; E_0, t\}; u, E_0\} \times C_1\{\sigma; u; E_1; t\}, \quad (33)$$

$$\frac{\partial C_1\{\sigma; E_0; E_1; t\}}{\partial t} = \int_0^\infty duC_2\{\sigma; E_0; E_1, u; t\}H\{\sigma; u\} + \int_0^\infty duC_1\{\sigma; E_0; u; t\}H_1\{\sigma; u; E_1\}. \quad (34)$$

We note from (25) that

$$H_1\{\sigma; E; E_1\} = -B(E)\delta(E - E_1) - \beta(E)\delta'(E - E_1) + 2\eta(E - E_1)q(E, E_1)\sigma(E - E_1) \quad (35)$$

and also that

$$H\{1; E_0\} = 0. \quad (36)$$

We cannot readily evaluate $H\{0; E_0\}$, however, for the requirement (10) entails a nonvanishing value for $d\sigma/dE$.

We can now find two equations for $K_1(E_0; E_1; t)$, by setting $\sigma = 1$ in (33) and (34): The backward equation is

$$\frac{\partial K_1(E_0; E_1; t)}{\partial t} = \int_0^\infty duH_1\{1; E_0; u\}K_1(u; E_1; t) = -B(E_0)K_1(E_0; E_1; t) - \beta(E_0)\frac{\partial K_1}{\partial E_0} + 2\int_0^{E_0} duq(E_0, u)K_1(u; E_1; t), \quad (37)$$

whereas the forward equation is

$$\frac{\partial K_1(E_0; E_1; t)}{\partial t} = \int_0^\infty duK_1(E_0; u; t)H_1\{1; u; E_1\} = -B(E_1)K_1(E_0; E_1; t) + \beta(E_1)\frac{\partial K_1}{\partial E_1} + 2\int_E^\infty duq(u, E_1)K_1(E_0; u; t). \quad (38)$$

Higher functional derivatives can be taken and evaluated at $\sigma = 1$, thus reproducing the hierarchy of equations given in Ref. 2. If on the other hand we evaluate the equation for $C, C_1, C_2, \cdots$ at $\sigma = 0$ (using integration by parts on the $d\sigma/dE$ term before setting $\sigma = 0$), we find a set of equations for P_N in which each $\partial P_N/\partial t$ involves P_{N+1} evaluated with one of its energy arguments equal to zero, again as in Ref. 2.

V. NUMERICAL EXAMPLES

A simple example for which $C\{\sigma; E_0, t\}$ can be found is that for which q is a constant,[8] $B(E) = qE$, and β is zero for all E. Then (28) becomes

$$\frac{\partial C}{\partial t} + qE_0C = q\int_0^{E_0} duC\{\sigma; u; t\}C\{\sigma; E_0 - u; t\}. \quad (39)$$

If we define the Laplace transform of C with respect to the variable E_0 by

$$\tilde{C}\{\tilde{\sigma}; \lambda; t\} = \int_0^\infty e^{-\lambda E}C\{\sigma; E; t\}\, dE \quad (40)$$

together with

$$\tilde{\sigma}(\lambda) = \int_0^\infty e^{-\lambda E}\sigma(E)\, dE, \quad (41)$$

[7] Reference 6, pp. 426–427.

[8] The example of $q = 1$ with β zero or not zero was treated from a less advanced point of view by W. T. Scott and G. E. Uhlenbeck, Phys. Rev. 62, 497 (1942).

we readily find from (39) that

$$\tilde{C}\{\tilde{\sigma};\lambda;t\} = \tilde{\sigma}(qt+\lambda)/[1-qt\tilde{\sigma}(qt+\lambda)]. \quad (42)$$

Using the usual Laplace inversion integral with $qt+\lambda$ replaced by λ, we get

$$C\{\sigma;E_0,t\} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} d\lambda \times \exp(\lambda E_0 - qtE_0)\frac{\tilde{\sigma}(\lambda)}{1-qt\tilde{\sigma}(\lambda)}, \quad (43)$$

where the integration is taken to the right of all singularities.

Letting $\sigma = z$ (except at $E_1 = 0$ and $\tilde{\sigma} = z/\lambda$), we readily find $G(z, E_0, t)$:

$$G(z,E_0,t) = \frac{z}{2\pi i}\int \frac{d\lambda \exp[(\lambda - qt)E_0]}{\lambda - qtz} = z\exp[(z-1)qtE_0]. \quad (44)$$

The probability of there being N particles is then

$$P(N,E_0t) = e^{-qtE_0}(qtE_0)^{N-1}/(N-1)!, \quad (45)$$

and the combinatorial moments are

$$K(M,E_0,t) = (M+qtE_0)(qtE_0)^{M-1}. \quad (46)$$

The mean is then

$$\langle N\rangle_{\mathrm{av}} = K(1,E_0,t) = 1+qtE_0$$

and the variance is

$$\langle N^2\rangle_{\mathrm{av}} - \langle N\rangle_{\mathrm{av}}^2 = K(2,E_0,t)+K(1,E_0,t) - K^2(1,E_0,t) = qtE_0, \quad (47)$$

showing that once E_0 and $\langle N\rangle_{\mathrm{av}}$ are large enough, the distribution is essentially Poisson.

When β is not zero, the Laplace transform $\tilde{C}$ must satisfy the equation

$$\frac{\partial\tilde{C}}{\partial t} - q\frac{\partial\tilde{C}}{\partial\lambda} + \beta(1-\lambda\tilde{C}) = q\tilde{C}^2. \quad (48a)$$

A dimensionless version of this equation may be obtained if we replace $\tilde{C}(q/B)^{\frac{1}{2}}$ by $\tilde{C}$, $\lambda(B/q)^{\frac{1}{2}}$ by λ, $E_0(q/B)^{\frac{1}{2}}$ by E_0, and $t(\beta q)^{\frac{1}{2}}$ by t. More simply, we can merely choose units of energy so that $\beta = q = 1$. Thus we write

$$\frac{\partial\tilde{C}}{\partial t} - \frac{\partial\tilde{C}}{\partial\lambda} + 1 - \lambda\tilde{C} = \tilde{C}^2, \quad (48b)$$

which becomes a Riccati equation if we replace t by the variable $t' = t+\lambda$. We find for the solution,

$$\tilde{C}\{\tilde{\sigma};\lambda;t\} = \left\{\tilde{\sigma}(\lambda+t)e^{\frac{1}{2}(\lambda+t)^2} - [(\lambda+t)\tilde{\sigma}(\lambda+t)-1]\int_\lambda^{\lambda+t} e^{\frac{1}{2}y^2}dy\right\} \times\left\{\lambda\tilde{\sigma}(\lambda+t)e^{\frac{1}{2}(\lambda+t)^2} - [(\lambda+t)\tilde{\sigma}(\lambda+t)-1] \times\left[e^{\frac{1}{2}\lambda^2} + \lambda\int_\lambda^{\lambda+t} e^{\frac{1}{2}y^2}dy\right]\right\}^{-1}. \quad (49)$$

The transform of G is then

$$\tilde{G}(z,\lambda,t) = \frac{ze^{\frac{1}{2}(\lambda+t)^2} - (\lambda+t)(z-1)\int_\lambda^{\lambda+t} e^{\frac{1}{2}y^2}dy}{\lambda z e^{\frac{1}{2}(\lambda+t)^2} - (\lambda+t)(z-1)\left[e^{\frac{1}{2}\lambda^2} + \lambda\int_\lambda^{\lambda+t} e^{\frac{1}{2}y^2}dy\right]}. \quad (50)$$

Expanding in powers of $(z-1)$, we get

$$\tilde{G}(z,\lambda,t) = \frac{1}{\lambda} + \frac{(\lambda+t)}{\lambda^2 e^{\frac{1}{2}t^2+\lambda t}}\sum_{M=1}^{\infty}(z-1)^M(-1)^{M-1} \times\left[\frac{\lambda e^{\frac{1}{2}t^2+\lambda t} - (\lambda+t) - \lambda(\lambda+t)\int_0^t e^{\frac{1}{2}x^2+x\lambda}dx}{\lambda e^{\frac{1}{2}t^2+\lambda t}}\right]^{M-1}, \quad (51)$$

whereas in powers of z, we find

$$\tilde{G}(z,\lambda,t) = \frac{\int_0^t e^{\frac{1}{2}x^2+x\lambda}dx}{1+\lambda\int_0^t e^{\frac{1}{2}x^2+x\lambda}dx} + \frac{e^{\frac{1}{2}t^2+t}}{(\lambda+t)^2\left[1+\lambda\int_0^t e^{\frac{1}{2}x^2+x\lambda}dx\right]^2} \times\sum_{N=1}^{\infty} z^N\left[1 - \frac{\lambda e^{\frac{1}{2}t^2+t\lambda}}{(\lambda+t)\left(1+\lambda\int_0^t e^{\frac{1}{2}x^2+x\lambda}dx\right)}\right]^{N-1}. \quad (52)$$

From (51) and the usual Laplace transform inversion formula we find

$$\langle N\rangle_{\mathrm{av}} = \frac{1}{2\pi i}\int d\lambda\, e^{\lambda E_0 - \lambda t - \frac{1}{2}t^2}\frac{(\lambda+t)}{\lambda^2} = e^{\frac{1}{2}t^2}[1+t(E_0-t)]; \quad 0<t<E_0 \quad (53)$$
$$= 0; \quad E_0 < t.$$

We also have

$$K(2,E_0,t) = \langle N^2\rangle_{\mathrm{av}} - \langle N\rangle_{\mathrm{av}} = \frac{-2}{2\pi i}\int\frac{d\lambda(\lambda+1)e^{\lambda E_0 - t^2 - 2\lambda t}}{\lambda^3} \times\left[\lambda e^{\frac{1}{2}t^2+\lambda t} - (\lambda+t) - \lambda(\lambda+t) \times\int_0^t dx\, e^{\frac{1}{2}x^2+x\lambda}\right]$$
$$= e^{-\frac{1}{2}t^2}(4t^2 - 2tE_0 - 2) + e^{-t^2}(4t^4 - 4t^3E_0 + t^2E_0^2 - 10t^2 + 4tE_0 + 2) + e^{-t^2}(2t^2E_0 - 4t^3 + 4t)^2\int_0^t dx\, e^{\frac{1}{2}x^2};$$
$$0 < t < \tfrac{1}{2}E_0$$

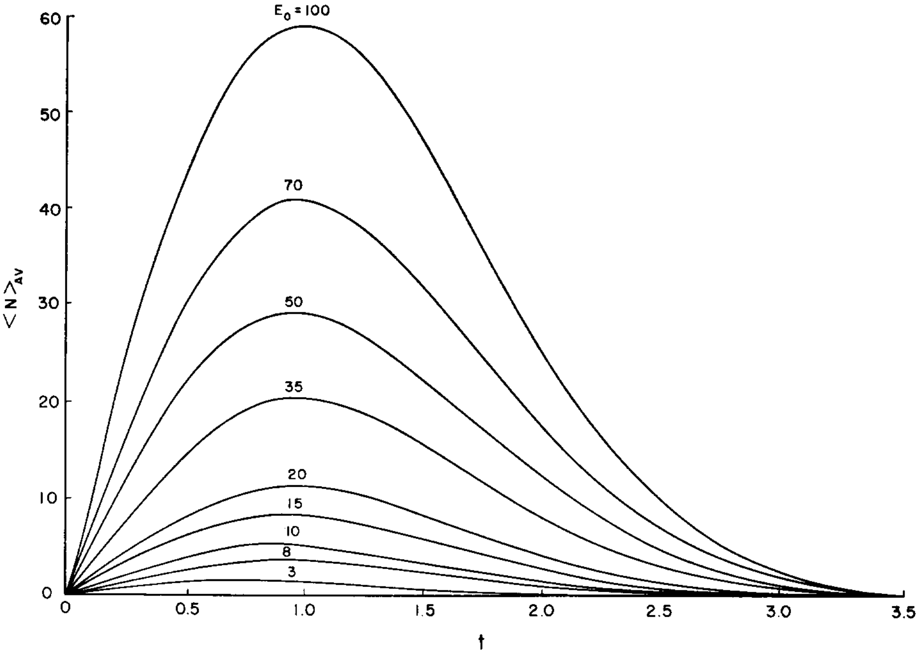


FIG. 1. The mean number of particles $\langle N\rangle_{\rm av}$ as a function of depth t in the shower described in the text, for different initial energies E_0. Dimensionless units are employed.

$$\begin{aligned}
&= e^{-\frac{1}{2}t^2}(4t^2 - 2tE_0 - 2) \\
&\quad + 2(1 - t^2)\exp\left(\tfrac{1}{2}E_0^2 - 2tE_0 + t^2\right) \\
&\quad + e^{-t^2}(2t^2E_0 - 4t^3 + 4t)\int_{2t-E_0}^{t} dx\, e^{\frac{1}{2}x^2}; \\
&\quad \tfrac{1}{2}E_0 < t < E_0 \\
&= 0; \quad E_0 < t. \qquad (54)
\end{aligned}$$

The dimensional forms for $\langle N\rangle_{\rm av}$ and $\langle N^2\rangle_{\rm av} - \langle N\rangle_{\rm av}$ may be obtained from (53) and (54) by replacing t by $t(\beta q)^{\frac{1}{2}}$, E_0t by qE_0t, and E_0 by $E_0(q/B)^{\frac{1}{2}}$. The results agree with Eqs. (29) and (30) of Ref. 8 except that the lower limit for the x integration in the case $\frac{1}{2}E_0 < t < E_0$ was mistakenly set at zero in the earlier reference.

Figures 1 and 2 show curves of $\langle N\rangle_{\rm av}$ and the variance divided $\langle N\rangle_{\rm av}$,

$$[\langle N^2\rangle_{\rm av} - \langle N\rangle_{\rm av}^2]/\langle N\rangle_{\rm av}.$$

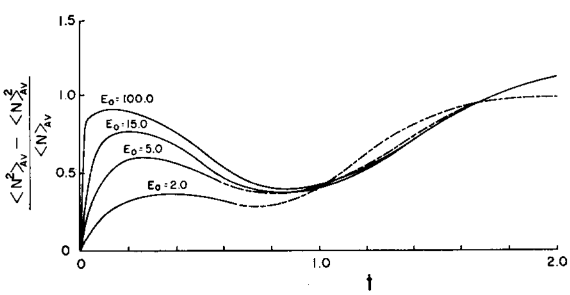


FIG. 2. The relative fluctuation or ratio of variance to mean, for the shower in the text, as a function of depth t for different initial energies E_0.

From (52) we can read out directly the Laplace transforms of each $P(N, E_0, t)$. Figures 3–10 show

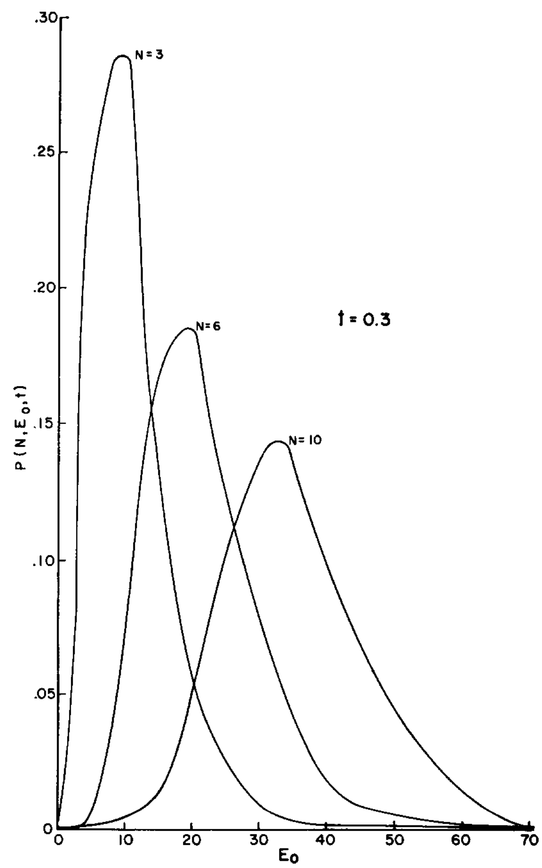


FIG. 3. The probability $P(N, E_0, t)$ of N particles at depth t when the incident energy is E_0, as a function of E_0 for $t = 0.3$ dimensionless units.

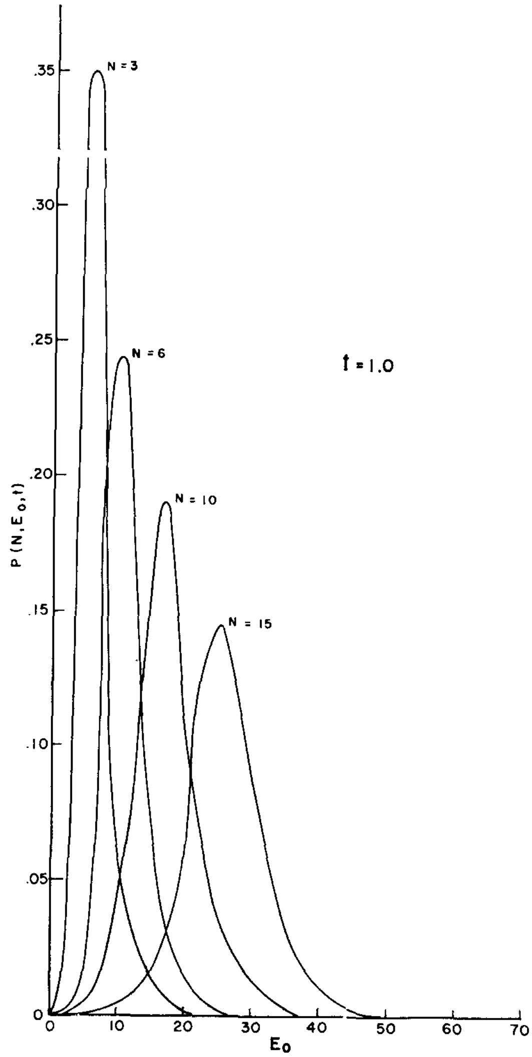


FIG. 4. Same as Fig. 3, for $t = 1.0$.

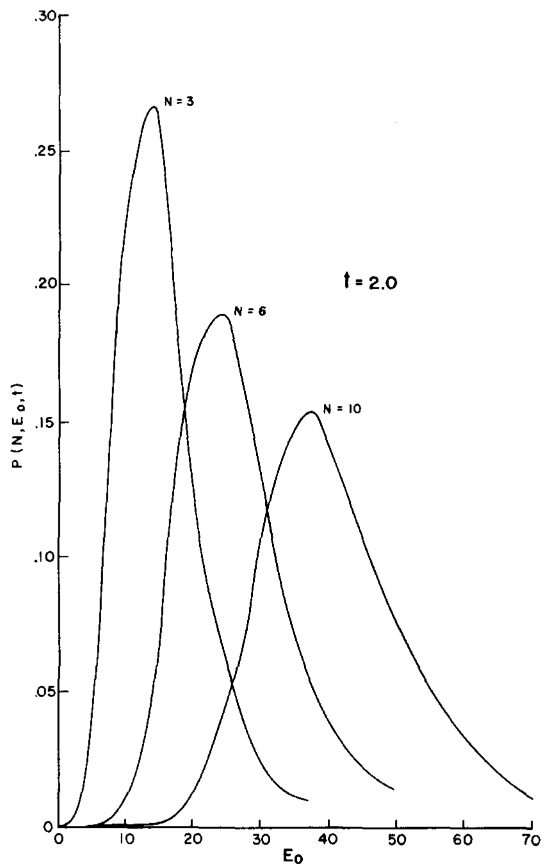


FIG. 5. Same as Fig. 3, for $t = 2.0$.

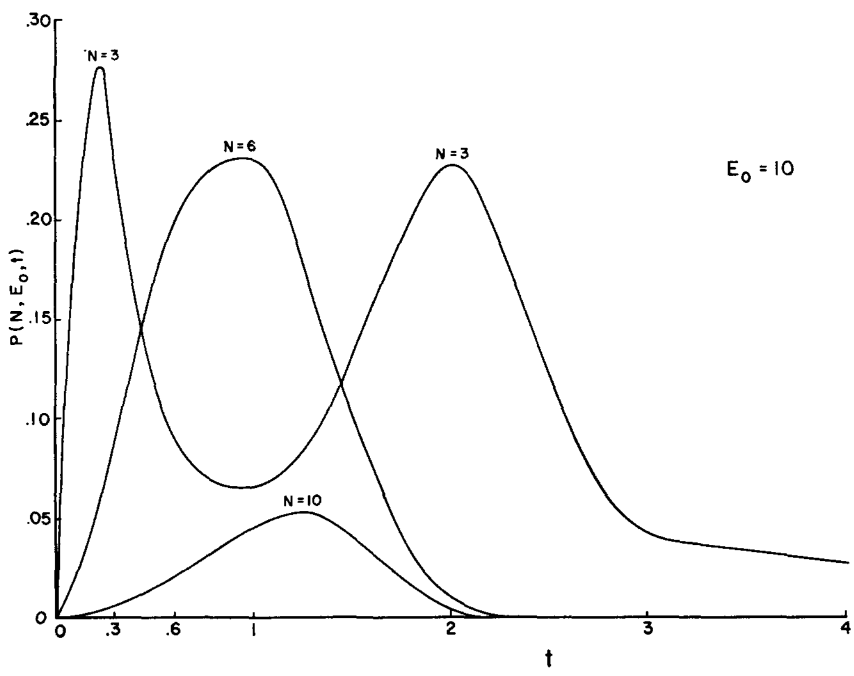


FIG. 6. $P(N, E_0, t)$ as a function of depth for $N = 3$, 6, and 10 at $E_0 = 10$.

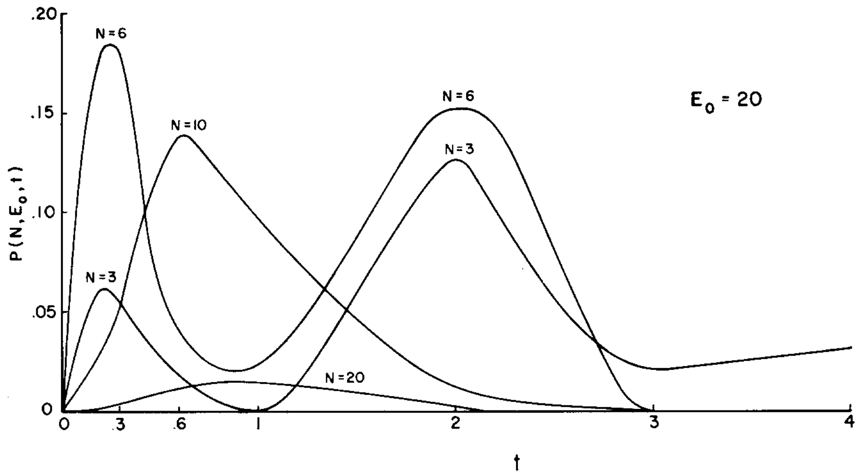


FIG. 7. Same as Fig. 6, for $N =$ 3, 6, 10, and 20 at $E_0 = 20$.

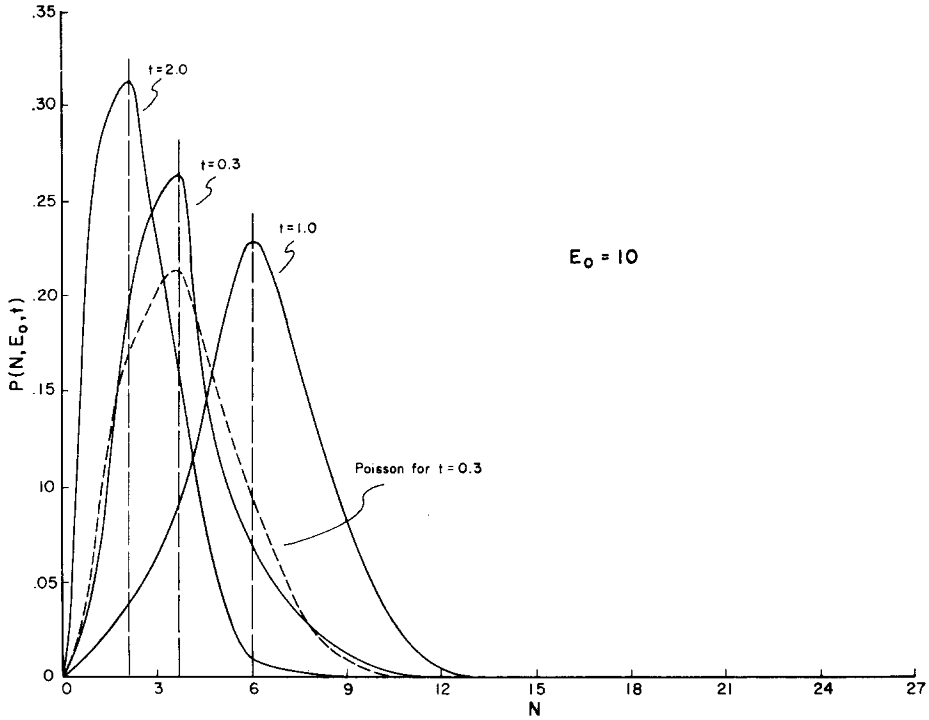


FIG. 8. $P(N, E_0, t)$ as a function of N for $t = 0.3$, 1.0 and 2.0, and $E_0 = 10$. The vertical lines represent the mean values $\langle N\rangle_{av}$. The dashed curve represents Poisson statistics for $t = 0.3$.

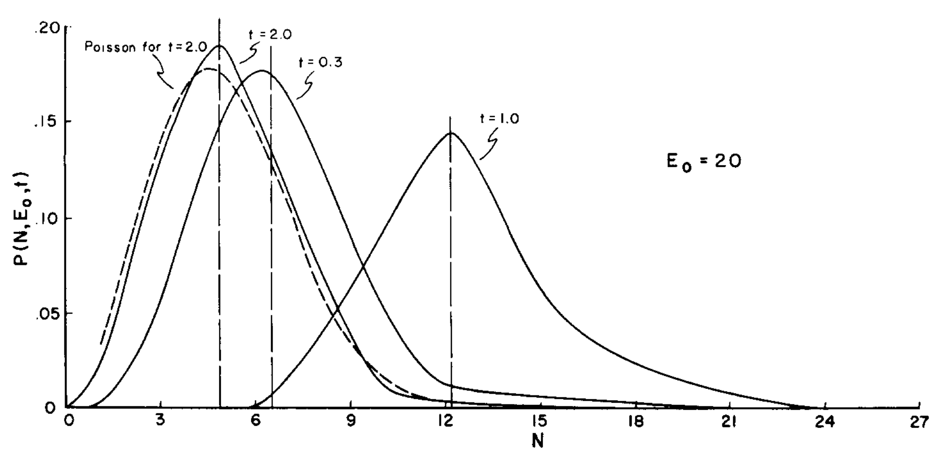


FIG. 9. Same as Fig. 8, for $E_0 = 20$. A Poisson distribution is drawn in for $t = 2.0$.

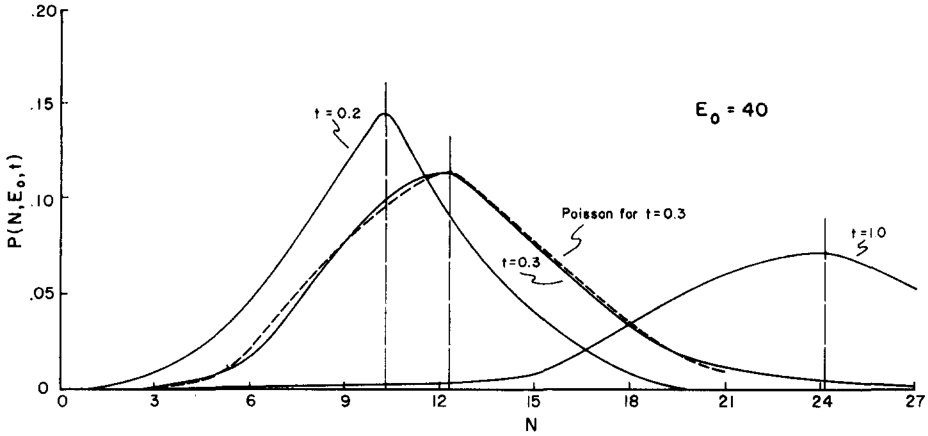


FIG. 10. Same as Fig. 8, for $E_0 = 40$. A Poisson distribution is shown for $t = 0.3$.

graphs of $P(N, E_0, t)$ for selected values of N, E_0, and t, calculated by the Laplace inversion techniques of Bellman *et al.*[9]

VI. A CLASS OF CHARACTERISTIC FUNCTIONALS

We have not been able to find the characteristic functional for any physically more realistic case such as that of the Furry[10] model with ionization in which β is a constant and $q(E, \mu) = 1/E$. However, it is possible to generate characteristic functionals that satisfy the Chapman–Kolmogoroff equation and may or may not have physical significance, by the following device.

Choose any invertible operation T that transforms an arbitrary function $\sigma(E)$ into a function $\tau(\mu)$ in some appropriate space. We write

$$T\{\sigma(E); \mu\} = \tau(\mu) \tag{55a}$$

and the inverse

$$T^{-1}\{\tau(\mu); E\} = \sigma(E). \tag{55b}$$

Next we choose a function $\phi(t)$ that has an inverse, so that if

$$s = \phi(t), \tag{56a}$$

then

$$t = \phi^{-1}(s). \tag{56b}$$

Now we define C by the relation

$$C\{\sigma; E_0, t\} = T^{-1}\{T\{\sigma(E); \phi[\phi^{-1}(\mu) + t]\} + at; E_0\}, \tag{57}$$

where a is a constant and T^{-1} operates on the variable μ in ϕ^{-1}.

To test (57) in (21), let us write

$$\sigma(E) = T^{-1}\{T\{\sigma'(E'); \phi[\phi^{-1}(\mu) + t']\} + at'; E\}.$$

When this expression is substituted in (57), we see by (55a) that the inner operation T yields

$$T\{\sigma'(E'); \phi[\phi^{-1}(\mu') + t']\} + at',$$

evaluated for $\mu' = \phi[\phi^{-1}(\mu) + t]$. We have

$$\phi\{\phi^{-1}(\phi[\phi^{-1}(\mu) + t]) + t'\} = \phi[\phi^{-1}(\mu) + t + t'],$$

so we obtain

$$\begin{aligned} &C\{C\{\sigma', E; t'\}; E_0; t\} \\ &= T^{-1}\{T\{\sigma'(E'); \phi[\phi^{-1}(\mu) + t + t']\} + a(t' + t); E_0\} \\ &= C\{\sigma'(E'); E_0; t + t'\}, \end{aligned}$$

thus showing that (57) satisfies (21).

It can be shown, for instance, that (49) is a special case of (57) with

$$\tau(\mu) = T\{\sigma; \mu\} = \frac{e^{\frac{1}{2}\mu^2}}{\mu - 1/\tilde{\sigma}(\mu)} - \int_0^\mu e^{\frac{1}{2}v^2}\, dy, \tag{58}$$

$$\begin{aligned} \sigma(E) &= T^{-1}\{\tau; E\} \\ &= \frac{1}{2\pi i}\int \frac{e^{\lambda E}\, d\lambda}{\lambda}\left[1 + \frac{e^{\frac{1}{2}\lambda^2}/\lambda}{\tau(\lambda) + \int_0^\lambda e^{\frac{1}{2}v^2}\, dy - e^{\frac{1}{2}\lambda^2}/\lambda}\right], \\ \phi(t) &= t, \\ \phi^{-1}(s) &= s, \\ a &= 0. \end{aligned}$$

The simpler case in which q is constant and $\beta = 0$ can be readily shown to be a case of (57) with

$$\begin{aligned} \tau(\mu) &= T\{\sigma; \mu\} = \frac{1}{\sigma(\mu)}, \\ T^{-1}\{\tau; E\} &= \frac{1}{2\pi\tau}\int \frac{e^{\lambda E}\, d\lambda}{\tau(\lambda)}, \\ \phi(t) &= qt, \\ \phi^{-1}(s) &= s/q, \\ a &= -q. \end{aligned} \tag{59}$$

VII. GENERALIZATION TO ELECTRON–PHOTON SHOWERS

When two kinds of particles are involved, the characteristic functional has two components. As

---

[9] R. E. Bellmann, H. H. Kagiwada, R. E. Kalaba, and M. C. Prestwood, *Invariant Imbedding and Time-Dependent Transport Processes* (American Elsevier Publishing Company, Inc., New York, 1964), Chap. 1.

[10] W. H. Furry, Phys. Rev. **52**, 569 (1937).

an example, we give the definition and the infinitesimal transformation generator for an electron–photon cascade, neglecting lateral spread. The master functions are as follows: The probability that a shower initiated by an electron of energy E_0 will generate N electrons of either sign of energies $E_1, \cdots, E_N$ and ν photons of energies $\mathcal{E}_1, \cdots, \mathcal{E}_\nu$ in thickness t is written as

$$P^e_{N\nu}(E_0; E_1, \cdots, E_N; \mathcal{E}_1, \cdots, \mathcal{E}_\nu; t), \tag{60a}$$

while the corresponding probability for a shower initiated by a photon of energy $\mathcal{E}_0$ is

$$P^\gamma_{N\nu}(E_0; E_1, \cdots E_N; \mathcal{E}_1; \cdots \mathcal{E}_\nu; t). \tag{60b}$$

The two components of the characteristic functional will each be functionals of two arbitrary functions $\sigma(E)$ and $\tau(\mathcal{E})$, as well as of one initial energy E_0 or $\mathcal{E}_0$ and the thickness t. We have, for instance,

$$\begin{aligned} C^e\{\sigma(E), \tau(\epsilon); E_0; t\} &= \sum_{N=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{1}{N!\nu!}\int_0^\infty dE_1 \cdots \int_0^\infty dE_N \\ &\times \int_0^\infty d\mathcal{E}_1 \cdots \int_0^\infty d\mathcal{E}_\nu \\ &\times \sigma(E_1)\cdots\sigma(E_N)\tau(\mathcal{E}_1)\cdots\tau(\mathcal{E}_\nu) \\ &\times P^e_{N\nu}(E_0; E_1, \cdots E_N; \mathcal{E}_1, \cdots, \mathcal{E}_\nu; t). \end{aligned} \tag{61}$$

The expression for C^γ differs by having $\mathcal{E}_0$ in place of E_0 and $P^\gamma{}_{N\nu}$ in place of $P^e{}_{N\nu}$.

Let us describe the set of physical processes involved in the shower by four functions: (a) The probability in dt that an electron or positron of energy E radiates a photon of energy $\mathcal{E}$ to $\mathcal{E} + d\mathcal{E}$ is $\pi(E, \mathcal{E})\, d\mathcal{E}\, dt$; (b) the probability in dt that a photon of energy $\mathcal{E}$ generates an electron–positron pair of energies E to $E + dE$ and $\mathcal{E} - E$ to $\mathcal{E} - E - dE$ is $\gamma(\mathcal{E}, E)\, dE\, dt = \gamma(\mathcal{E}, \mathcal{E} - E)\, dE\, dt$; (c) the probability that photon of energy $\mathcal{E}$ will undergo a Compton scattering in dt and have its energy reduced to $\mathcal{E}' + d\mathcal{E}'$ is $c(\mathcal{E}, \mathcal{E}')\, d\mathcal{E}'\, dt$; (d) an electron or positron of energy E will lose energy $\beta(E)\, dt$ in dt.

We readily find for the generator of infinitesimal transformations

$$\begin{aligned} H^e\{\sigma(E), \tau(\mathcal{E}); E_0\} &= -\beta(E_0)\sigma'(E_0) - \sigma(E_0)\int_0^{E_0} dE\, \pi(E, \mathcal{E}) \\ &+ \int_0^{E_0} dE\, \pi(E, \mathcal{E})\tau(\mathcal{E})\sigma(E - \mathcal{E}), \end{aligned} \tag{61a}$$

$$\begin{aligned} H^\gamma\{\sigma(E), \tau(\mathcal{E}); \mathcal{E}_0\} &= -\tau(E_0)\int_0^{E_0} dE\, \gamma(\mathcal{E}_0, E) \\ &- \tau(\mathcal{E}_0)\int_0^{\mathcal{E}_0} d\mathcal{E}\, c(\mathcal{E}_0, \mathcal{E}) + \int_0^{\mathcal{E}_0} d\mathcal{E}\, c(\mathcal{E}_0, \mathcal{E})\sigma(\mathcal{E}) \\ &+ \int_0^{\mathcal{E}_0} dE\, \gamma(\mathcal{E}_0, E)\sigma(E)\sigma(\mathcal{E}_0 - E). \end{aligned} \tag{62b}$$

The infinitesimal transformations are then

$$\begin{aligned} C^e\{\sigma(E), \tau(E); E_0; \Delta t\} &= \sigma(E_0) + \Delta t H^e\{\sigma(E), \tau(\mathcal{E}); E_0\}, \end{aligned} \tag{63}$$

$$C^\gamma\{\sigma(E), \tau(\mathcal{E}); \mathcal{E}_0; \Delta t\} = \tau(\mathcal{E}_0) + \Delta t\, H^\gamma\{\sigma(E), \tau(\mathcal{E}); \mathcal{E}_0\}. \tag{64}$$

The Chapman–Kolmogoroff relation now becomes two equations:

$$\begin{aligned} C^e\{\sigma(E), \tau(\mathcal{E}); E_0; t + t'\} &= C^e\{C^e\{\sigma(E''), \tau(\mathcal{E}''); E; t\}, \\ &\qquad C^\gamma\{\sigma(E'), \tau(\mathcal{E}'); \mathcal{E}; t\}; E_0; t'\}, \end{aligned} \tag{65a}$$

$$\begin{aligned} C^\gamma\{\sigma(E), \tau(\mathcal{E}); \mathcal{E}_0; t + t'\} &= C^\gamma\{C^e\{\sigma(E''), \tau(\mathcal{E}''); E; t\}; \\ &\qquad C^\gamma\{\sigma(E'), \tau(\mathcal{E}'); \mathcal{E}, t\}; \mathcal{E}_0, t'\}. \end{aligned} \tag{65b}$$

The backward and forward equations corresponding to (28) and (30) become the two coupled pairs:

$$\begin{aligned} \partial C^e\{\sigma, \tau; E_0; t\}/\partial t &= H^e\{C^e\{\sigma, \tau; E; t\}, C^\gamma\{\sigma, \tau; \mathcal{E}; t\}; \mathcal{E}_0\}, \end{aligned} \tag{66a}$$

$$\begin{aligned} \partial C^\gamma\{\sigma, \tau; \varepsilon_0, t\}/\partial t &= H^\gamma\{C^e\{\sigma, \tau; E; t\}, C^\gamma\{\sigma, \tau; \varepsilon, t\}; \varepsilon_0\}, \end{aligned} \tag{66b}$$

and

$$\begin{aligned} \frac{\partial C^e\{\sigma, \tau; E_0; t\}}{\partial t} &= \int_0^\infty du\, C^e_{1;0}\{\sigma, \tau; E_0; u; t\}H^e\{\sigma, \tau; u\} \\ &+ \int_0^\infty dv\, C^e_{0;1}\{\sigma, \tau; E_0; v; t\}H^\gamma\{0, \tau; v\}, \end{aligned} \tag{67a}$$

$$\begin{aligned} \frac{\partial C^e\{\sigma, \tau; E_0; t\}}{\partial t} &= \int_0^\infty du\, C^\gamma_{1;0}\{\sigma, \tau; E_0, u, t\}H^e\{\sigma, \tau; u\} \\ &+ \int_0^\infty dv\, C^\gamma_{0;1}\{\sigma, \tau; E_0, v, t\}H^\gamma\{\sigma, \tau; v\}. \end{aligned} \tag{67b}$$

VIII. CONCLUSION

Many relations among probabilities of interest for multiplicative processes involving a parameter like energy or age can be readily and elegantly derived by use of the characteristic functional given in this paper. While its actual use in computing distributions for physically interesting cases must await further developments, such as that of appropriate variational principles or extensions of the Green's function method, the relations among the various quantities can be exhibited with some transparency by this method. Thus it may constitute a modest step toward the solution of a class of hitherto intractable problems.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Expansion Method for Nonlinear Boundary-Value Problems*

JOSÉ CANOSA†
General Electric Company, Pleasanton, California

(Received 20 February 1967)

A homogeneous nonlinear boundary-value problem, which reduces to the Helmholtz equation when the nonlinearity is removed, is solved by an expansion method using as a basis the eigenfunctions of the linear Helmholtz equation. The nonlinear differential equation is reduced to a nonlinear algebraic system in the expansion coefficients, which can be easily solved with any desired degree of accuracy. It is found that with only three terms in the eigenfunction expansion, a satisfactory agreement with the exact numerical solution of the problem is obtained, even for strongly nonlinear cases. Solutions are presented both in one-dimensional slab and cylindrical geometries. It is also shown that the method can be applied to inhomogeneous problems.

1. INTRODUCTION

THE aim of this paper is to present a simple method for solving a class of nonlinear boundary-value problems, which reduce to the Helmholtz equation when the nonlinearities are removed. The problem presented arose in connection with the determination of the distribution of the energy released in a nuclear reactor as a result of a power excursion.

Explicit solutions are given in one-dimensional geometries for a homogeneous boundary value problem with a certain type of nonlinearity. It is then shown that the method can also be used to solve inhomogeneous boundary value problems.

2. NONLINEAR HELMHOLTZ EQUATION

Ergen[1] has shown recently that the distribution of the energy released in a nuclear reactor as a result of a power excursion is given by the solution of the following nonlinear boundary-value problem:

$$\begin{gathered}\nabla^2 y(x) + (1 - cy)y = 0,\\ y(S) = 0,\end{gathered} \tag{1}$$

where S is the domain boundary, and c is a given positive constant. We have called Eq. (1) a nonlinear Helmholtz equation because if the nonlinearity is removed ($c = 0$), the classical Helmholtz equation is obtained. In this paper, we only consider one-dimensional and symmetric domains. Proofs of the existence and uniqueness of the solution of nonlinear elliptic boundary-value problems are very difficult to obtain. For equations of the general type (1), Courant and Hilbert[2] show that solutions exist if

$$|(1 - cy)y| \leq N, \tag{2}$$

i.e., when the nonlinear function in Eq. (1) is bounded in the domain.

From the physical nature of the problem, we conjecture that Eq. (1) has a solution which is unique, positive, and bounded. Using this assumption, the following plausible argument indicates that the solution has an explicit upper bound. By the boundedness assumption, the solution of (1) must have at least a regular maximum in the domain, excluding irregular distributions. Let one such maximum occur at $\bar{x}$, then

$$\begin{gathered}y(\bar{x}) = M, \quad dy(\bar{x})/dx = 0,\\ \nabla^2 y(\bar{x}) = a[d^2y(\bar{x})/dx^2] = M(cM - 1) < 0.\end{gathered} \tag{3}$$

In slab geometry, $a \equiv 1$; for cylinders, $a = 1$ for $\bar{x} \neq 0$ and $a = 2$ for $\bar{x} = 0$; for spheres $a = 1$ for $\bar{x} \neq 0$ and $a = 3$ for $\bar{x} = 0$. This is because if the maximum is at the origin,

$$\lim_{x\to 0} \frac{1}{x}\frac{dy(x)}{dx} = \frac{d^2y(0)}{dx^2},$$

by L'Hopital's rule. But for $x \neq \bar{x}$ we also have

$$\nabla^2 y(x) = y(cy - 1) < 0, \tag{4}$$

because $y < M$. Hence, there can be no minimum for any x, because this requires

$$dy(x)/dx = 0, \quad \nabla^2 y(x) = a(d^2y/dx^2) > 0,$$

in contradiction with Eq. (4). Therefore, the only maximum of the solution occurs at $\bar{x}$. By symmetry, this maximum must be at the center of the domain. From Eq. (3), we have

$$y(x) < y(\bar{x}) = M < 1/c. \tag{5}$$

In what follows, an expansion method for the solution of Eq. (1) is presented. The expansion uses as a basis the eigenfunctions of the linear Helmholtz equation associated with the problem.

* This work was supported by the U.S. Atomic Energy Commission under Subcontract C-276 with the Phillips Petroleum Company.
† Present address: Intenational Business Machines Scientific Center, Palo Alto, California.
[1] W. K. Ergen, Trans. Am. Nucl. Soc. 8, 221 (1965).
[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1962), Vol. II, p. 369.

3. EIGENFUNCTION EXPANSION METHOD

In what follows, we present the method in one-dimensional slab and cylindrical geometries. The solution to the boundary-value problem (1) is sought in the form of a series

$$y(x) = \sum_{\nu=1}^{\infty} A_\nu \varphi_\nu(x), \tag{6}$$

where φ_ν satisfy the linear Helmholtz equation associated with (1),

$$\begin{aligned} \nabla^2 \varphi_\nu + B_\nu^2 \varphi_\nu &= 0, \\ \varphi_\nu(S) &= 0. \end{aligned} \tag{7}$$

As the problem under study has symmetry, we choose

$$\begin{aligned} \varphi_\nu(x) &= \cos\,(2\nu - 1)(\pi/2R)x, \\ B_\nu &= (2\nu - 1)(\pi/2R), \\ \nu &= 1, 2, 3, \ldots, \end{aligned} \tag{8}$$

for slab geometry, and

$$\varphi_\nu = J_0(B_\nu r), \quad B_\nu R = \nu\text{th positive zero of } J_0(x), \quad \nu = 1, 2, 3, \ldots, \tag{9}$$

for cylindrical geometry. $J_0(x)$ is the Bessel function of the first kind, and R is the slab half width or the cylinder radius. Substituting Eq. (6) into Eq. (1) and using (7), one gets

$$\sum_{\nu=1}^{\infty} (1 - B_\nu^2) A_\nu \varphi_\nu = c \sum_{\nu=1}^{\infty} A_\nu \varphi_\nu \sum_{k=1}^{\infty} A_k \varphi_k. \tag{10}$$

Equation (10) is the fundamental starting point of this analysis. We define the j-mode approximation as that in which only the first j summands are kept in the summations of (10). In detail, the nonlinear term then becomes

$$\begin{aligned} c \sum_{\nu=1}^{j} A_\nu \varphi_\nu \sum_{k=1}^{j} A_k \varphi_k = c[&A_1^2\varphi_1^2 + A_2^2\varphi_2^2 + \cdots + A_j^2\varphi_j^2 \\ &+ 2A_1A_2\varphi_1\varphi_2 + \cdots + 2A_1A_j\varphi_1\varphi_j + 2A_2A_3\varphi_2\varphi_3 \\ &+ \cdots + 2A_2A_j\varphi_2\varphi_j + \cdots + 2A_{j-1}A_j\varphi_{j-1}\varphi_j]. \end{aligned} \tag{11}$$

Using the orthogonality properties of the eigenfunctions (7), the space variable is now eliminated from (10) by multiplying it by $\varphi_l(x)$ (slab geometry) or $x\varphi_l(x)$ (cylindrical geometry) and integrating over the domain. In this way, we obtain a coupled nonlinear algebraic system in the expansion coefficients A_ν:

$$\begin{aligned} &b_1^{11}A_1^2 + b_1^{22}A_2^2 + \cdots + b_1^{jj}A_j^2 + 2b_2^{11}A_1A_2 + \cdots + 2b_j^{11}A_1A_j \\ &\qquad + 2b^{1,2,3}A_2A_3 + \cdots + 2b^{1,2,j}A_2A_j + \cdots + 2b^{1,j-1,j}A_{j-1}A_j = c_1A_1, \\ &b_2^{11}A_1^2 + b_2^{22}A_2^2 + \cdots + b_2^{jj}A_j^2 + 2b_1^{22}A_1A_2 + \cdots + 2b^{1,2,j}A_1A_j \\ &\qquad + 2b_3^{22}A_2A_3 + \cdots + 2b_j^{22}A_2A_j + \cdots + 2b^{2,j-1,j}A_{j-1}A_j = c_2A_2, \\ &\qquad\qquad \cdots \qquad \cdots \qquad \cdots \\ &b_j^{11}A_1^2 + b_j^{22}A_2^2 + \cdots + b_j^{jj}A_j^2 + 2b^{1,2,j}A_1A_2 + \cdots + 2b_1^{jj}A_1A_j \\ &\qquad + 2b^{2,3,j}A_2A_3 + \cdots + 2b_2^{jj}A_2A_j + \cdots + 2b^{j,j-1,j}A_{j-1}A_j = c_jA_j. \end{aligned} \tag{12}$$

The notation in system (12) is as follows:

$$b_m^{ll} = \frac{1}{R}\int_{-R}^{R} \varphi_l^2\varphi_m\,dx, \qquad b^{l,m,n} = \frac{1}{R}\int_{-R}^{R} \varphi_l\varphi_m\varphi_n\,dx,$$

$$c_j = (1 - B_j^2)/c, \tag{13}$$

for slab geometry, and

$$b_m^{ll} = \frac{1}{R^2}\int_0^R x\varphi_l^2\varphi_m\,dx, \quad b^{l,m,n} = \frac{1}{R^2}\int_0^R x\varphi_l\varphi_m\varphi_n\,dx,$$

$$c_j = \{(1 - B_j^2)[J_1(B_jR)]^2\}/2c, \tag{14}$$

for cylindrical geometry; here, $J_1(x)$ is the Bessel function of the first kind, of order one. The eigenfunctions and eigenvalues used were defined in Eqs. (8) and (9).

In slab geometry, the integrals appearing in (13) can always be determined in closed form. In particular, the b_m^{ll} are given compactly in the form:

$$b_m^{ll} = \frac{(-1)^{m+1}}{\pi}\left[\frac{1}{3 + 4(l-1) - 2m} + \frac{2}{2m-1} - \frac{1}{1 + 4(l-1) + 2m}\right]. \tag{15}$$

In cylindrical geometry, the integrals in (14) cannot be obtained in closed form, but are readily obtained numerically by Simpson's rule. In Table I, we give the first 10 integrals (14) required for the three-mode approximation; they were calculated using a 21-point Simpson's rule.

One should note that once the b coefficients of (12) have been calculated, they remain the same for all boundary-value problems (1). The only new data for each problem are the coefficients c_j in the right-hand side of Eq. (12), which can be readily obtained from

TABLE I. Bessel integrals (14).

b_1^{11}	9.744×10^{-2}
b_1^{22}	3.642×10^{-2}
b_1^{33}	2.279×10^{-2}
b_2^{11}	1.804×10^{-2}
b_3^{11}	-1.758×10^{-3}
b_2^{22}	7.481×10^{-3}
b_2^{33}	4.587×10^{-3}
b_3^{22}	1.513×10^{-2}
b_3^{33}	6.434×10^{-3}
$b^{1,2,3}$	9.937×10^{-3}

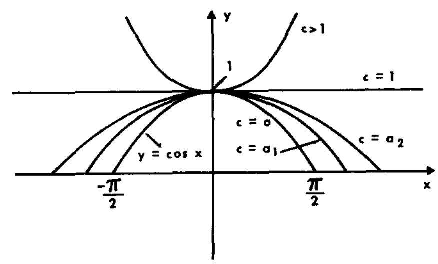


FIG. 1. Qualitative plot of the solutions of the initial value problem (16) in slab geometry, $0 < a_1 < a_2 \cdots < 1$.

Eqs. (13) and (14). It should also be noticed that, to the order of the modal approximation used, the nonlinear terms have been treated rigorously; that is, no linearization of terms is involved in any order of approximation.

4. DISCUSSION OF RESULTS

For slab geometry, the solution of Eq. (1) can be expressed in terms of elliptic functions, and for cylindrical geometry, the solution does not seem to be given in terms of tabulated functions.[3] McKinney solved numerically the following initial value problem associated with (1):

$$\begin{aligned} \nabla^2 y + (1 - cy)y &= 0, \\ y(0) = 1, \quad dy(0)/dx &= 0, \end{aligned} \tag{16}$$

for slab and cylindrical geometries. For clarity, in Fig. 1 we give a qualitative plot of the solutions of (16) in slab geometry. Using his numerical results, the corresponding boundary-value problem (1) was generated using for each value of c, the domain size at which the initial value problem solution vanished. In this way, for each value of c, the solutions of the initial and boundary-value problems are identical and can be compared directly.

When only the first (fundamental) eigenfunction is kept in the expansion (6), system (12) reduces to

$$b_1^{11} A_1^2 = c_1 A_1, \tag{17}$$

or

$$A_1 = c_1 / b_1^{11}. \tag{18}$$

The results obtained from the "exact" numerical solution of (1) and from the one-mode treatment are given in Table II. For the strongly nonlinear cases, $c \to 1$, the distribution (Fig. 1) becomes markedly different from the linear distribution, $c = 0$. For slab and cylindrical geometries, the one-mode approximation is a fair approximation to the exact solution when $c \lessapprox 0.9$ and $c \lessapprox 0.7$, respectively.

In the two-mode approximation, one lets $j = 2$ in (12) and it becomes:

$$\begin{aligned} b_1^{11} A_1^2 + b_1^{22} A_2^2 + 2 b_2^{11} A_1 A_2 &= c_1 A_1, \\ b_2^{11} A_1^2 + b_2^{22} A_2^2 + 2 b_1^{22} A_1 A_2 &= c_2 A_2. \end{aligned} \tag{19}$$

In the three-mode approximation, we let $j = 3$ in (12) and this becomes:

$$\begin{aligned} b_1^{11} A_1^2 + b_1^{22} A_2^2 + b_1^{33} A_3^2 + 2 b_2^{11} A_1 A_2 + 2 b_3^{11} A_1 A_3 \\ + 2 b^{1,2,3} A_2 A_3 = c_1 A_1, \\ b_2^{11} A_1^2 + b_2^{22} A_2^2 + b_2^{33} A_3^2 + 2 b_1^{22} A_1 A_2 + 2 b^{1,2,3} A_1 A_3 \\ + 2 b_3^{22} A_2 A_3 = c_2 A_2, \\ b_3^{11} A_1^2 + b_3^{22} A_2^2 + b_3^{33} A_3^2 + 2 b^{1,2,3} A_1 A_2 + 2 b_1^{33} A_1 A_3 \\ + 2 b_2^{33} A_2 A_3 = c_3 A_3, \end{aligned} \tag{20}$$

and so forth. Nonlinear algebraic systems such as

TABLE II. One-mode approximation vs exact solution.

c, Nonlinear parameter	Domain size R Slab	Cyl.	Max value of y at domain center: Slab Exact	Slab 1-mode approx	Cylinder Exact	Cylinder 1-mode approx	Integral of solution over domain: Slab Exact	Slab 1-mode approx	Slab % Diff.	Cylinder Exact	Cylinder 1-mode approx	Cylinder % Diff.
0.3	1.821	2.724	1	1.008	1	1.025	2.344	2.329	0.6	2.434	2.393	1.7
0.5	2.078	3.039	1	1.010	1	1.035	2.709	2.671	1.4	2.812	2.714	3.5
0.7	2.503	3.536	1	1.025	1	1.064	3.336	3.255	2.4	3.462	3.241	6.4
0.9	3.511	4.653	1	1.049	1	1.126	4.957	4.681	5.6	5.135	4.525	11.9
0.99	5.777	7.064	1	1.102	1	1.234	9.108	8.105	11.0	9.426	7.530	20.1

[3] A. W. McKinney, General Electric Report GEAP-3450 (1960).

TABLE III. Two-mode approximation vs exact solution.

c, Nonlinear parameter	Domain size R Slab	Cyl.	Max value of y at domain center: Slab Exact	Slab 2-mode approx	Cylinder Exact	Cylinder 2-mode approx	Integral of solution over domain: Slab Exact	Slab 2-mode approx	Slab % Diff.	Cylinder Exact	Cylinder 2-mode approx	Cylinder % Diff.
0.7	2.503	3.536	1	1.000	1	0.989	3.336	3.330	0.2	3.462	3.452	0.4
0.9	3.511	4.653	1	0.992	1	0.964	4.957	4.932	0.5	5.135	5.067	1.3
0.99	5.777	7.064	1	0.967	1	0.890	9.108	8.935	1.9	9.426	9.005	4.5

(19) and (20) can easily be solved by the Newton–Raphson iteration method.[4] It is noted that appropriate initial guesses for A_1, A_2, and A_3 have been found to be respectively A_1, as given by (18), $-A_1/10$ (the minus sign because the second mode must necessarily lead to a flattening of the fundamental mode distribution), and $A_1/100$. Convergence of the Newton–Raphson method was obtained in this way after only three iterations in all cases studied. Systems (19) and (20) were programmed for solution by a small electronic computer, although, if necessary, they can be solved with a desk calculator.

In Table III, the results obtained with the two-mode approximation are given, together with the exact numerical results. The two-mode approximation is already in fairly good agreement with the exact answer, in particular for slab geometry. As expected, the agreement worsens for the more nonlinear cases, where the flattening of the distribution becomes more pronounced.

In Table IV, we give the results obtained from the three-mode approximation. The agreement is much improved relative to the two-mode approximation. This agreement is quite satisfactory even for the strongest nonlinear case studied, $c = 0.99$, both in slab and cylindrical geometries. The excellent accuracy obtained with three modes for the integral of the distribution is a further check that the exact distribution is uniformly well approximated over all the domain, and not only at its center.

5. INHOMOGENEOUS NONLINEAR BOUNDARY-VALUE PROBLEMS

The method developed in Sec. 3 can be applied without difficulty to inhomogeneous problems. Consider the simplest case:

$$\nabla^2 y + (1 - cy)y = -f(x), \quad y(S) = 0. \tag{21}$$

We now expand formally

$$y(x) = \sum_{\nu=1}^{\infty} A_\nu \varphi_\nu(x), \qquad f(x) = \sum_{\nu=1}^{\infty} B_\nu \varphi_\nu(x), \tag{22}$$

and follow the same procedure as in Sec. 3. The only difference in the result of the treatment is that, instead of a *homogeneous* nonlinear algebraic system, such as (12), we obtain an *inhomogeneous* system whose homogeneous part is identical to (12).

6. CONCLUSION

An eigenfunction expansion method for the solution of nonlinear boundary value problems has been developed. This treatment, in which the nonlinear terms are taken rigorously into account, is especially useful for the solution of symmetric problems, where a few-mode approximation is expected to give good results. In the examples presented, a two-mode approximation leads already to results in fairly good agreement with the exact solution.

TABLE IV. Three-mode approximation vs exact solution.

c, Nonlinear parameter	Domain size R Slab	Cyl.	Max value of y at domain center: Slab Exact	Slab 3-mode approx	Cylinder Exact	Cylinder 3-mode approx	Integral of solution over domain: Slab Exact	Slab 3-mode approx	Slab % Diff.	Cylinder Exact	Cylinder 3-mode approx	Cylinder % Diff.
0.7	2.503	3.536	1	1.000	1	1.003	3.336	3.333	0.1	3.462	3.463	0.0
0.9	3.511	4.653	1	1.003	1	1.011	4.957	4.951	0.1	5.135	5.123	0.2
0.99	5.777	7.064	1	1.007	1	1.045	9.108	9.166	0.6	9.426	9.305	1.3

[4] F. B. Hildebrand, *Introduction to Numerical Analysis* (McGraw-Hill Book Company, Inc., New York, 1956), p. 451.

Interpolation Method for the Many-Body Problem*

L. Kijewski and J. K. Percus
Courant Institute of Mathematical Sciences, New York University, New York

(Received 12 May 1967)

Variational principles for lower bounds to the energy, or free energy for $T > 0°$, of many-body systems are obtained in a form requiring density matrix minimization subject to certain model restrictions. The latter restrict the domain in which the density matrices can vary, and only utilize the energy—or free energy—for the model Hamiltonian H_M. Increasingly accurate bounds are obtained as the model system begins to resemble the system of interest, and the behavior of the error as $H - H_M$ approaches zero is shown by two examples based upon the Ising model. Coupling the lower bound principle for the free energy with the standard Gibbs–Bogoliubov upper bound principle results in bounds on generalized susceptibility as well.

1. INTRODUCTION

One of the most standard, yet powerful, techniques for obtaining the ground state energy $E_0(H)$ for a system with Hamiltonian H is the Rayleigh–Ritz principle, in which one varies the wave function ψ so as to minimize $\int \psi^* H\psi\, d\tau$. For *two*-body interactions in an N-body system,

$$H = \sum_1^N T(i) + \frac{1}{2}\sum_{i\neq j} v(i,j) \tag{1.1}$$

and ground state wavefunction (anti) symmetric in the particle coordinates; this is equivalent to

$$E_0(H) = \frac{N}{2}\min_{f^{(2)}} \operatorname{Tr} H^{(2)} f^{(2)}, \tag{1.2}$$

where

$$H^{(2)} = T(1) + T(2) + (N-1)v(1,2), \tag{1.3}$$

provided the normalized reduced density matrix $f^{(2)}(1'2' \mid 1\,2)$ comes from integrating (or summing) the N-body density matrix of some real system;

$$f^{(2)}(1'2' \mid 1\,2) = \int\cdots\int \sum \lambda_i \psi_i(1'2'3\cdots N) \times \psi_i^*(1\,2\,3\cdots N)\, d3\cdots dN \tag{1.4}$$

($\lambda_i \geq 0$, $\sum \lambda_i = 1$). If the restriction (1.4) is disregarded, then an exact minimization of (1.2) over four-argument functions $f^{(2)}$ will result in a lower bound to $E_0(H)$.

If we impose further conditions on $f^{(2)}$ which are implied by (1.4),[1] then the lower bound of (1.2) will be improved. The question we now ask ourselves is: Can we in fact go all the way to $E_0(H)$, without use of a wavefunction, by restricting $f^{(2)}$ tightly enough? For this purpose, we may divide the conditions which $f^{(2)}$ must satisfy into two classes: kinematic and model-dependent. The distinction is principally psychological. Kinematic conditions will refer to those such as normalization, positivity, symmetry:

$$\operatorname{Tr} f^{(2)} = 1, \tag{1.5}$$

$f^{(2)}$ is *nonnegative*, as an operator

$$f^{(2)}(1'2' \mid 1\,2) = f^{(2)}(2'1' \mid 2\,1) = \pm f^{(2)}(1'2' \mid 2\,1) \quad \text{for} \begin{Bmatrix}\text{E.B.}\\ \text{F.D.}\end{Bmatrix} \text{statistics,}$$

as well as more sophisticated conditions on eigenvalues, relations between $f^{(1)}$ and $f^{(2)}$, etc. A model-dependent condition arises from the existence of a model Hamiltonian H_M whose ground state $E_0(H_M)$ is known. Then, according to (1.2), any legitimate $f^{(2)}$, i.e., one satisfying (1.4), must also satisfy

$$C_M(f^{(2)}) \equiv \operatorname{Tr} H_M^{(2)} f^{(2)} - \frac{2}{N} E_0(H_M) \geq 0. \tag{1.6}$$

In fact, it can be shown that if (1.6) holds for all possible H_M (all two-body interaction Hamiltonians), then $f^{(2)}$ must have the form (1.4),[2] and the minimum of (1.2) is exact. The second of (1.5) is incidentally a special model condition, that in which $T_M(i) = 0$, $\langle 1'2' |\, v_M \,| 1\,2\rangle = \phi(1'2')\phi^*(1\,2)$ for any two-body function ϕ.

Let us suppose that the kinematic conditions (1.5) are always imposed. There are then two ways in which models may be employed. First, one at a time, in the form

$$\frac{N}{2}\operatorname{Tr} H^{(2)} f_M^{(2)} \geq E_0(H) \geq \frac{N}{2}\min_{C_M(f^{(2)})\geq 0} \operatorname{Tr} H^{(2)} f^{(2)} \tag{1.7}$$

with the upper bound presupposing knowledge of $f_M^{(2)}$. Equation (1.7) can now be improved by using

---

* Supported in part by Air Force Office of Scientific Research, Contract AF-AFOSR-945-65, and the U.S. Atomic Energy Commission, Contract AT(30-1)-1480.

[1] See C. Garrod and J. K. Percus, J. Math. Phys. 5, 1756 (1964); A. J. Coleman, Rev. Mod. Phys. 35, 668 (1963); C. Garrod, Phys. Fluids 9 1764 (1966); E. Feenberg, J. Math. Phys. 6, 658 (1965).

[2] C. Garrod and J. K. Percus, Ref. 1.

the best model of some available class:

$$\frac{N}{2}\min_M \operatorname{Tr} H^{(2)} f_M^{(2)} \geq E_0(H)$$
$$\geq \frac{N}{2}\max_M \min_{C_M(f^{(2)})\geq 0} \operatorname{Tr} H^{(2)} f^{(2)}. \quad (1.8)$$

Second, the set of models may be used as interpolation points, each restricting the domain in which $f^{(2)}$ may vary. Thus,

$$E_0(H) \geq \frac{N}{2} \min_{\{C_M(f^{(2)})\geq 0\}} \operatorname{Tr} H^{(2)} f^{(2)}, \quad (1.9)$$

{ } indicating the whole set of restrictions. Equation (1.9) always gives a higher (or equal) lower bound.

In this paper, we first investigate the fashion in which Eqs. (1.8) and (1.9) yield increasingly accurate results as the model systems become closer to the system in question. We then extend the technique to thermal equilibrium, restricting attention for definiteness to Ising models, and to systems involving weak change sin known situations: application of a weak magnetic field. The ratio of accuracy achieved to information required is gratifyingly high.

2. APPROACH TO EXACT SOLUTION

Since the variational principle (1.9) is expressed by means of linear equalities and inequalities, the error in E_0 in the vicinity of the exact answer cannot be a quadratic function of the parameters being varied, as in the ordinary upper bound principle. This indeed is one of the disadvantages of the technique. To gain some feeling as to how the error varies, we consider the effect of a single model-dependent restriction in which the model can be chosen arbitrarily close to the system being analyzed.

We henceforth restrict our major attention to lattice gases, defined by the condition that the particles are confined to the points of a spatial lattice, each of which can be occupied by at most one particle. We use the equivalent Ising model terminology, attributing a "spin" σ $(=\sigma_z) = \pm 1$ to each lattice site. Consider then the Ω-site Ising model with magnetic field, given by

$$H = -J\sum_{\langle ij\rangle} \sigma_z(i)\sigma_z(j) - B\sum_1^\Omega \sigma_x(j), \quad (2.1)$$

σ_x and σ_z being the usual Pauli spin matrices and $\langle ij\rangle$ denoting a sum over nearest neighbors only.[3] Since the interaction is symmetric among nearest neighbor pairs, the ground state can be chosen as symmetric as well. Thus, rather than the full lattice gas pair density matrix, we require only the 4×4 nearest-neighbor spin density matrix $f^{(2)}(\sigma_1\sigma_2 \mid \sigma_1'\sigma_2')$. Equations (1.2) and (1.3) become

$$E_0(H) = \frac{\Omega}{2}\min_{f^{(2)}} \operatorname{Tr} H^{(2)} f^{(2)}, \quad (2.2)$$

$$H^{(2)} = -ZJ\sigma_z(1)\sigma_z(2) - B[\sigma_x(1) + \sigma_x(2)], \quad (2.3)$$

where Z is the number of nearest neighbors, and periodic boundaries are assumed.

To start with, let us carry out (2.2) subject only to normalization and nonnegativity, denoting the resulting density matrix by $f_0^{(2)}$. This is just a spin-$\frac{1}{2}$ two-particle problem, and so in a spin-1 plus spin-0 representation

$$H^{(2)} = \begin{bmatrix} -ZJ & -B\sqrt{2} & 0 & 0 \\ -B\sqrt{2} & ZJ & -B\sqrt{2} & 0 \\ 0 & -B\sqrt{2} & -ZJ & 0 \\ 0 & 0 & 0 & ZJ \end{bmatrix}, \quad (2.4)$$

the indices referring to configurations $(++)$, $(+-)+(-+)/\sqrt{2}$, $(--)$, $(+-)-(-+)/\sqrt{2}$. $H^{(2)}$ is readily diagonalized, and we then find

$$f_0^{(2)} = \psi_0^T \psi_0, \quad (2.5)$$

$$\psi_0 = \left(\tfrac{1}{2}(1+\gamma)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}(1-\gamma)^{\frac{1}{2}}, \tfrac{1}{2}(1+\gamma)^{\frac{1}{2}}, 0\right),$$

where

$$\gamma = ZJ/(Z^2J^2 + 4B^2)^{\frac{1}{2}},$$

while

$$E_0(H) \geq \frac{\Omega}{2} E_0(H^{(2)}) = -\Omega(B^2 + \tfrac{1}{4}Z^2J^2)^{\frac{1}{2}}, \quad (2.6)$$

a lower bound for all Ω. Since the ground-state correlation structure changes with Ω [as opposed, e.g., to $H = -J\sum \boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j) - B\sum \sigma_x(j)$, $J > 0$, in which the ground state has each $\sigma_x(j) = 1$ independently] (2.6) is an equality only for $\Omega = 2$.

Now as model system, we choose a replica of the system in question, but with different parameters:

$$H_M = -J'\sum_{\langle ij\rangle} \sigma_z(i)\sigma_z(j) - B'\sum_1^\Omega \sigma_x(j), \quad (2.7)$$

and assume that $E_0(H_M)$ is known. Then the model restriction (1.6) may be characterized loosely by its strength. If Eq. (1.6) is already satisfied by $f_0^{(2)}$,

$$\text{WEAK:} \quad \tfrac{1}{2}\Omega \operatorname{Tr} H_M^{(2)} f_0^{(2)} \geq E_0(H_M), \quad (2.8)$$

we speak of a weak restriction, and are free to disregard it. In the present case, this condition may be written as

$$\cos(\theta - \theta') \leq E_0(H_M) \Big/ \left[\frac{\Omega}{2} E_0(H_M^{(2)})\right], \quad (2.9)$$

[3] Some properties of this model are discussed by M. E. Fisher in J. Math. Phys. 4, 124 (1963).

where $\tan\theta = 2B/ZJ$, $\tan\theta' = 2B'/ZJ'$, and will hold only for θ' sufficiently far from θ. It is for example satisfied by the field-free model $B' = 0$ (reducing there to $\cos\theta \le 1$) which is thereby useless. If (2.8) is not satisfied, the restriction (1.6) will reduce the domain of variation of $f^{(2)}$, and since only linear equalities and inequalities are involved, $f^{(2)}$ must then lie on the boundary of the domain. Thus, (1.6) must be imposed as an equality. Introducing the Lagrange parameter λ, we must now solve

$$E_0(H) \ge E_0 = \frac{\Omega}{2}\min_{f^{(2)}} \operatorname{Tr} f^{(2)}(H^{(2)} - \lambda H_M^{(2)}) + \lambda E_0(H_M),$$

where

$$\frac{\Omega}{2}\operatorname{Tr} f^{(2)} H_M^{(2)} = E_0(H_M). \tag{2.10}$$

A further subdivision then occurs, according to whether the resulting $f^{(2)}$ is a pure state $\psi^T\psi$ or not. If it is, we may speak of a moderate restriction, and (2.10) becomes

$$\text{MODERATE:}\quad E_0(H) \ge E_0 = \frac{\Omega}{2}E_0(H^{(2)} - \lambda H_M^{(2)}) + \lambda E_0(H_M), \tag{2.11}$$

where
$$\langle\psi| H_M^{(2)} |\psi\rangle = \frac{2}{\Omega}E_0(H_M),$$

the second condition determining the value of λ. In the example of Eqs. (2.3) and (2.7), $H^{(2)} - \lambda H_M^{(2)}$ has the same form as $H^{(2)}$. Thus, the results (2.5) and (2.6) can be used, and yield after brief computation

$$E_0 = \frac{\Omega}{2}\cos(\theta_M - |\theta - \theta'|)E_0(H^{(2)}),$$

where

$$\cos\theta_M = E_0(H_M)\Big/\left[\frac{\Omega}{2}E_0(H_M^{(2)})\right], \tag{2.12}$$

using the notation of (2.9), and $0 \le \theta_M < \pi/2$. If (2.12) were to continue being relevant for H_M in the vicinity of H, then on setting

$$\theta' - \theta = \Delta,\quad E_0(H)\Big/\left[\frac{\Omega}{2}E_0(H^{(2)})\right] = \cos\theta_0,$$

and expanding about $H_M - H \sim 0$, $\Delta \sim 0$, (2.12) takes on the form

$$\frac{E_0}{E_0(H)} = 1 + \left[\frac{\langle H_M - H\rangle}{\langle H\rangle} - \frac{\langle H_M^{(2)} - H^{(2)}\rangle}{\langle H^{(2)}\rangle}\right] + |\tan\theta_0\Delta| \tag{2.13}$$

(where expectations are, e.g., in the respective eigenstates of H and $H^{(2)}$) with its characteristic broken line maximum. (See Fig. 1.)

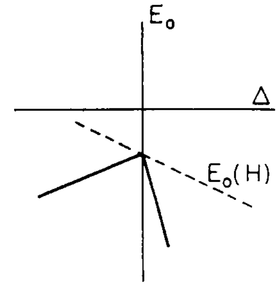


FIG. 1. Lower bound to ground state energy using a single model restriction.

As H_M approaches H, we would expect $f^{(2)}$ to approach its correct value as well, and this is certainly not in the form of a pure state. In fact, if $H_M = H$, we see that (2.10) has the solution $\lambda = 1$, together with any $f^{(2)}$ satisfying the second equation—which of course includes the correct $f^{(2)}$. A single model condition will simply not be enough for a unique determination of $f^{(2)}$ at this stage. At any rate, there must be a point in the shrinking of the domain of $f^{(2)}$ as H_M approaches H, where the minimum of (2.10) occurs at a degenerate $f^{(2)}$:

$$f^{(2)} = \sum_{i,j=1}^{4}\alpha_{ij}\psi_i^T\psi_j, \tag{2.14}$$

$$\operatorname{Tr}\alpha = 1,\quad \alpha \text{ nonnegative}.$$

When the restriction is this strong, λ is fixed simply by

$$\text{STRONG:}\quad E_0(H^{(2)} - \lambda H_M^{(2)}) \text{ is degenerate}, \tag{2.15}$$

and the restriction is satisfied by (2.14) with some α providing that when ψ_1 and ψ_2 are chosen to diagonalize the matrix $\langle\psi_i| H_M^{(2)} |\psi_j\rangle$, we have

$$\langle\psi_1| H_M^{(2)} |\psi_1\rangle \ge \frac{2}{\Omega}E_0(H_M) \ge \langle\psi_2| H_M^{(2)} |\psi_2\rangle. \tag{2.16}$$

If (2.15) and (2.16) hold, the resulting

$$E_0 = \tfrac{1}{2}\Omega\langle\psi_2| H^{(2)} - \lambda H_M^{(2)} |\psi_2\rangle + \lambda E_0(H_M)$$

is lower than that for a pure state variation since [assuming that the analog of (2.8) fails] the weaker restriction

$$\langle\psi_2| H_M^{(2)} |\psi_2\rangle \ge \frac{2}{\Omega}E_0(H_M) - \epsilon,$$

for some ϵ, is being used. Conversely, as soon as the right-hand side of (2.16) begins to fail, the pure state minimum takes over.

In the example of Eqs. (2.3) and (2.7), the full set of eigenvalues is given by $\pm ZJ$, $\pm(Z^2J^2 + 4B^2)^{\frac{1}{2}}$. Hence, $E_0(H^{(2)} - \lambda H_M^{(2)})$ becomes degenerate when $B - \lambda B' = 0$, and we have

$$\psi_1 = (1\,0\,0\,0),\quad \psi_2 = (0\,0\,1\,0),$$

$$\langle\psi_i| H_M^{(2)} |\psi_j\rangle = \begin{pmatrix} -ZJ' & 0 \\ 0 & -ZJ' \end{pmatrix}.$$

But then Eq. (2.16) can never obtain for $H_M \neq H$, and the strong regime does not set in. In general, when there are several model restrictions, a rather complicated set of mixed strengths may have to be investigated.

Finally, we may be more explicit and reduce our example to the special case of 3 sites; $\Omega = 3$, $Z = 2$. Then

$$H = -2J\bar{\sigma}_z^2 + \tfrac{3}{2}J - 2B\bar{\sigma}_x, \tag{2.17}$$

where $\bar{\boldsymbol{\sigma}} = \frac{1}{2}(\boldsymbol{\sigma}(1) + \boldsymbol{\sigma}(2) + \boldsymbol{\sigma}(3))$ is the sum of three spin $\frac{1}{2}$'s. The full space decomposes into one spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ spaces, only the first of which gives rise to the lowest eigenvalue. Hence, using the spin-$\frac{3}{2}$ representation of $\bar{\boldsymbol{\sigma}}$, we have

$$H = \begin{bmatrix} -3J & -B\sqrt{3} & 0 & 0 \\ -B\sqrt{3} & J & -2B & 0 \\ 0 & -2B & J & -B\sqrt{3} \\ 0 & 0 & -B\sqrt{3} & -3J \end{bmatrix}, \tag{2.18}$$

with ground state

$$E_0(H) = J[-1 - b - 2(1 - b + b^2)^{\frac{1}{2}}], \qquad b = B/J. \tag{2.19}$$

For small b, Eq. (2.6) for $\Omega = 3$ yields $J[-3 - \frac{3}{2}b^2 \cdots]$, as opposed to the exact $J[-3 - \frac{3}{4}b^2 \cdots]$ above. Next, employing a model described by $b' = B'/J'$ (a multiplicative factor has no effect upon the model restriction), Eq. (2.11) or (2.12) yields

$$E_0 = -J\left(\frac{1 + bb'}{1 + b'^2}\right)[1 + b' + 2(1 - b' + b'^2)^{\frac{1}{2}}] - J\sqrt{2}\,\frac{|b - b'|}{1 + b'^2}\,|(1 - b' + b'^2)^{\frac{1}{2}} - b' - 1| \tag{2.20}$$

with the typical form of Fig. 1. Numerically, one finds, e.g., for $b = \frac{3}{8}$ that

$$E_0 = -J\left[3\tfrac{1}{8} + \begin{matrix} (\frac{32}{73}\sqrt{2} - \frac{160}{511})(b' - \frac{3}{8}),\ b' > \frac{3}{8} \\ (\frac{32}{73}\sqrt{2} + \frac{160}{511})(\frac{3}{8} - b'),\ b' < \frac{3}{8} \end{matrix}\right]$$

in the vicinity of the maximum, illustrating the possible large asymmetries—the nonzero slope of the dotted reflection plane in Fig. 1.

3. EXTENSION TO STATISTICAL MECHANICS

Presumably, there is no ultimate distinction between kinematics and dynamics, since the former relates to possible system configurations, the latter to that which actually exists. However, under any circumstances, an artificial separation may prove a useful computational tool, as it has already proved. The kinematical quantities are then those common to the set of models available, and the remaining explicit dynamics is to be solved exactly. Precisely this procedure is available for states in thermal equilibrium, where the configuration common to all models is effectively specified by the combination of pair density and entropy.

The statistical mechanics of an isothermal ($T = 1/k\beta$), isochoric (volume Ω), petit (particle number N) ensemble is determined by the Helmholtz free energy

$$F(H) = -\frac{1}{\beta}\ln \operatorname{Tr} e^{-\beta H}, \tag{3.1}$$

which reduces to the ground state $E_0(H)$ as $\beta \to \infty$. Now the density matrix variational principle

$$E_0(H) = \min \operatorname{Tr} H\Gamma,$$

where

$$\operatorname{Tr}\Gamma = 1,\ \Gamma \text{ nonnegative}, \tag{3.2}$$

for H of the form (1.1) can be expressed in terms of the reduced two-body density matrix $f^{(2)}$ alone. But in (1.2), the corresponding free energy principle

$$F(H) = \min (E - TS) = \min \operatorname{Tr}\left(H\Gamma + \frac{1}{\beta}\Gamma \ln \Gamma\right) \tag{3.3}$$

involves the highly nonlinear $\Gamma \ln \Gamma$, and thus, in terms of density matrices, requires not one but all. The remedy, as we have indicated, is direct and trivial but effective. We regard the pair $(f^{(2)}, s)$ consisting of a pair density together with a single scalar, the entropy per particle (in units of Boltzmann's constant), as the kinematic object to be varied. Then Eq. (3.3) becomes

$$F(H) = \min N\left(\tfrac{1}{2}\operatorname{Tr} H^{(2)}f^{(2)} - \frac{s}{\beta}\right) \tag{3.4}$$

if $(f^{(2)}, s)$ is derivable from some N-body Γ.

Equation (3.4) now poses two problems. First, that of restricting $f^{(2)}$ to possible reductions of Γ—a problem which we have considered—and second, that of determining the maximum entropy allowable for a given $f^{(2)}$. Since the zero-temperature models relevant to the first problem are limits (as $T \to 0°$, or equivalently, as the scale λ of the Hamiltonian λH goes to ∞) of the finite temperature models relevant to the second problem, it suffices to consider models H_M at the common desired temperature. Suppose then that the free energies $\{F(H_M)\}$ of the class of models $\{H_M\}$ are known. Any legitimate pair $(f^{(2)}, s)$ must satisfy, according to Eq. (3.3),

$$s/\beta \leq \tfrac{1}{2}\operatorname{Tr} H_M^{(2)}f^{(2)} - \frac{1}{N}F(H_M) \tag{3.5}$$

for each H_M, and hence,

$$s/\beta \leq \min_{\{H_M\}} \left[\tfrac{1}{2} \operatorname{Tr} H_M^{(2)} f^{(2)} - \frac{1}{N} F(H_M) \right]. \quad (3.6)$$

But (3.4) reads $F(H) \geq \min N [\frac{1}{2} \operatorname{Tr} H^{(2)} f^{(2)} - (s/\beta)]$ when $(f^{(2)}, s)$ is varied over a class including those derivable from an N-body Γ. Hence, (3.6) can be used to eliminate s/β, yielding

$$F(H) \geq F = N \min_{f^{(2)}} \max_{\{H_M\}} \left[\tfrac{1}{2} \operatorname{Tr} (H^{(2)} - H_M^{(2)}) f^{(2)} + \frac{1}{N} F(H_M) \right], \quad (3.7)$$

where $f^{(2)}$ is varied over any domain including $\Gamma^{(N)}$-derivable ones.

As in our ground-state energy discussion, a weaker result (with a lower lower-bound) is obtained by using one model at a time, and then finding the largest F[4]:

$$F(H) \geq F = N \max_{\{H_M\}} \min_{f^{(2)}} \left[\tfrac{1}{2} \operatorname{Tr} (H^{(2)} - H_M^{(2)}) f^{(2)} + \frac{1}{N} F(H_M) \right]. \quad (3.8)$$

In either event, in Eq. (3.7) or (3.8), $f^{(2)}$ may be restricted by any number of additional model conditions, from all to none. The strictly statistical effects have already been approximated by the model upper bound to s, and this approximation is irretrievable. Thus, even if $f^{(2)}$ is guaranteed to come from an N-body Γ, (3.8) becomes the approximation

$$F(H) \geq F = \max_{\{H_M\}} [E_0(H - H_M) + F(H_M)] \quad (3.9)$$

with, however, the interesting and useful property of converting the statistical problem to one of finding an exact ground state. If $f^{(2)}$ is only restricted to two-body validity, i.e., only positive and normalized, Eq. (3.8) instead becomes

$$F(H) \geq F = \max_{\{H_M\}} \left[\frac{N}{2} E_0(H^{(2)} - H_M^{(2)}) + F(H_M) \right], \quad (3.10)$$

the weakest possible form. Of course, replacement of (3.10) by (3.7) generally raises the lower bound; the form of the result does not simplify in this case.

To gain some estimate of how strong our new conditions are, we may go to the $T \to 0^\circ$ limit. The weakest result (3.10) becomes

$$E_0(H) \geq E_0 = \max_{\{H_M\}} \left[\frac{N}{2} E_0(H^{(2)} - H_M^{(2)}) + E_0(H_M) \right]. \quad (3.11)$$

For the intrinsically nonlinear statistical ensemble, multiplication of H_M by a scalar does yield new information. Thus, using the model sequence $\{\lambda H_M\}$ at fixed H_M in (3.11),

$$E_0(H) \geq E_0 = \max_{\lambda} \left[\frac{N}{2} E_0(H^{(2)} - \lambda H_M^{(2)}) + E_0(\lambda H_M) \right], \quad (3.12)$$

which is equivalent to (1.8) and more convenient since the classification (2.8, 2.11, 2.15) of model restrictions does not have to be spelled out.

When (3.9) is analogously reduced to zero temperature,

$$E_0(H) \geq E_0 = \max_{\lambda} [E_0(H - \lambda H_M) + E_0(\lambda H_M)], \quad (3.13)$$

it merely becomes an expression, but a potentially useful one of the concavity of the minimum eigenvalue. The strongest result (3.7) at $T = 0^\circ$,

$$E_0(H) \geq E_0 = N \min_{f^{(2)}} \max_{\{H_M\}} \left[\tfrac{1}{2} \operatorname{Tr} (H^{(2)} - H_M^{(2)}) f^{(2)} + \frac{1}{N} E_0(H_M) \right] \quad (3.14)$$

appears weaker than (1.9), since the strongest subsidiary condition is simply added to the quantity to be minimized—i.e., one subtracts the entropy and not some function of it—but generalizing H_M to the class $\{\lambda H_M\}$, (1.9) is reproduced.

4. PERTURBATIONAL UPPER AND LOWER BOUNDS

Under many circumstances, the system under discussion is close to some standard system in a quantitative fashion specified by a perturbation parameter. The standard system may then serve itself as a "model." Application of a perturbation can represent a physical situation, or merely be an artifice. For example, a computational technique which yields a poor $f^{(2)}$ will also yield poor expectations $\langle Q \rangle$ for other than the energy. It may then be useful to construct $\langle Q \rangle$ by applying Q via an appropriate field. For this purpose, if λ is a generalized field strength and Q the corresponding generalized moment, and if we then define

$$M_Q(\lambda) = \langle Q \rangle_{H=H_0-\lambda Q}, \quad (4.1)$$

we of course have

$$\langle Q \rangle_{H_0} = M_Q(0). \quad (4.2)$$

But the alternative to (4.1),

$$M_Q(\lambda) = -\frac{\partial}{\partial \lambda} F(H_0 - \lambda Q) \equiv -\frac{\partial F_\lambda}{\partial \lambda} \quad (4.3)$$

[4] This form may be compared with the standard Bogoliubov principle as given, e.g., by M. D. Girardeau, J. Chem. Phys. **40**, 899 (1964).

and the further alternatives to (4.2),

$$M_Q(0) = \lim_{0<\lambda\to 0} \frac{1}{\lambda}(F_0 - F_\lambda) = \lim_{0<\lambda\to 0} \frac{1}{\lambda}(F_{-\lambda} - F_0), \tag{4.4}$$

where these limits coincide, require, as desired, only the free energy. Thus, if F_0 is known and we have an approximation $\bar{F}_\lambda \leq F_\lambda$, (4.4) gives respective upper and lower bounds to $\langle Q\rangle$; if $\bar{F}_\lambda \geq F_\lambda$, the bounds are reversed.

When λ represents a physical field with moment Q, Eq. (4.4) is relevant if the system has a permanent moment. If a permanent moment is only induced by the removal of field-free degeneracy, then (4.4) yields instead the two permanent moments $M_Q^\pm$, respectively. Each is in general bounded from one side, but symmetry relations may relate $M_Q^\pm$ and so allow two-sided bounds again.

If a permanent moment does not exist, but only an induced moment, then $\langle Q\rangle_{H_0} = 0$ and we are interested in the generalized susceptibility

$$\mu_Q(\lambda) = \frac{\partial M_Q(\lambda)}{\partial\lambda} = -\frac{\partial^2}{\partial\lambda^2} F(H_0 - \lambda Q). \tag{4.5}$$

Since $\langle Q\rangle_{H_0} = 0$, the initial susceptibility can be written as

$$\mu_Q(0) = \lim_{\lambda\to 0} \frac{2}{\lambda^2}(F(0) - F(\lambda)), \tag{4.6}$$

and only a single bound is available. Away from the origin, (4.5) is required, but the differential λ-dependence can be reduced. To do so, we note that since

$$F(H_0 - \lambda Q) = -\frac{1}{\beta} \ln \operatorname{Tr} \exp[-\beta(H_0 - \lambda Q)],$$

then

$$-F_\lambda'(H_0 - \lambda Q) = \operatorname{Tr} Q \exp[-\beta(H_0 - \lambda Q)] / \operatorname{Tr} \exp[-\beta(H_0 - \lambda Q)]$$

and

$$\begin{aligned}-F_\lambda''(H_0 - \lambda Q) &= \operatorname{Tr}\int_0^\beta \exp\left[-\frac{\alpha}{2}(H_0 - \lambda Q)\right] Q \\ &\quad\times \exp[-(\beta-\alpha)(H_0 - \lambda Q)]Q \\ &\quad\times \exp\left[-\frac{\alpha}{2}(H_0 - \lambda Q)\right] d\alpha / \operatorname{Tr}\exp[-\beta(H_0 - \lambda Q)] \\ &\quad- \beta(\operatorname{Tr} Q \exp[-\beta(H_0 - \lambda Q)] / \operatorname{Tr}\exp[-\beta(H_0 - \lambda Q)])^2.\end{aligned}$$

Thus, if

$$Q(\alpha) \equiv \exp[-\alpha(H_0 - \lambda Q)] Q \exp[\alpha(H_0 - \lambda Q)], \tag{4.7}$$

we have

$$-F_\lambda''(H_0 - \lambda Q) = \int_0^\beta \left(\left\langle Q\left(\frac{\alpha}{2}\right)Q^+\left(\frac{\alpha}{2}\right)\right\rangle + \left\langle Q\left(\frac{\alpha}{2}\right)Q^+\left(\frac{\alpha}{2}\right)\right\rangle\right) d\alpha. \tag{4.8}$$

Parenthetically, we remark that (4.8) establishes the concavity—nonnegativity of the right-hand side—of the free energy with respect to a linear parameter. Now, to avoid complications which are not specifically germane to the general approach, we will deal with classical statistical mechanics hereafter. Equation (4.8) hence reduces to

$$\mu_Q(\lambda) = \langle Q^2\rangle_{H_0-\lambda Q} - (\langle Q\rangle_{H_0-\lambda Q})^2 = \min_\gamma \langle (Q-\gamma)^2\rangle_{H_0-\lambda Q}, \tag{4.9}$$

just another moment problem, with, e.g., all the advantages thereof at $\lambda = 0$.

The $\lambda = 0$ bounds are not generally available at finite λ for the moment and susceptibility, although of course they are for the corresponding finite increment quantities:

$$\Delta F_Q(\lambda) \equiv \frac{1}{\lambda}(F(H_0) - F(H_0 - \lambda Q)),$$

$$\Delta^2 F_Q(\lambda) \equiv \frac{1}{\lambda^2}(2F(H_0) - F(H_0 - \lambda Q) - F(H_0 + \lambda Q)). \tag{4.10}$$

Further from the nonnegativity of (4.8),

$$F(0) = F(\lambda) - \lambda F'(\lambda) + \frac{\lambda^2}{2} F''(\theta\lambda)$$

for

$$0 \leq \theta \leq \lambda,$$

so that

$$F(0) \leq F(\lambda) - \lambda F'(\lambda),$$

or

$$-F'(\lambda) \geq \frac{F(0) - F(\lambda)}{\lambda}, \tag{4.11}$$

yielding a weak bound if $F(\lambda)$ is replaced by an upper bound $\bar{F}(\lambda)$.

Let us now investigate the quality of the bounds on free energy available in the perturbation domain. We define

$$H = H_0 + \Delta, \tag{4.12}$$

regarding H_0 as a model, and Δ as a first-order infinitesimal. Then a direct expansion of the free energy $F(H) = -\beta^{-1} \ln \operatorname{Tr} \exp(-\beta H)$ for (4.12) yields

$$F(H) = F(H_0) + \bar{\Delta} - \frac{\beta}{2}\langle(\Delta - \bar{\Delta})^2\rangle + \frac{\beta^2}{6}\langle(\Delta - \bar{\Delta})^3\rangle \cdots, \quad (4.13)$$

where $\bar{\Delta} \equiv \langle\Delta\rangle$ and all expectations are at $H = H_0$. This is to be compared with the upper bound given by (3.3) for $\Gamma(H_0) = \exp(-\beta H_0)/\operatorname{Tr}\exp(-\beta H_0)$ as variational density,

$$F(H) \leq F(H_0) + \bar{\Delta}. \quad (4.14)$$

Equation (4.14) is of course exact to first order. Under many circumstances, however, model information is trivially available when some model parameter varies as well, notably the temperature—or equivalently the energy scale—in the present case. Thus,

$$F(H) \leq \min\,[F(\gamma H_0) + \langle\Delta\rangle_{\gamma H_0} - (\gamma - 1)\langle H_0\rangle_{\gamma H_0}],$$

the minimum occurring at

$$\langle\Delta H_0\rangle - \langle\Delta\rangle\langle H_0\rangle = (\gamma - 1)(\langle H_0^2\rangle - \langle H_0\rangle^2),$$

so that

$$F(H) \leq F(\gamma H_0) + \frac{\langle H_0^2\rangle_{\gamma H_0}\langle\Delta\rangle_{\gamma H_0} - \langle H_0\rangle_{\gamma H_0}\langle\Delta H_0\rangle_{\gamma H_0}}{\langle H_0^2\rangle_{\gamma H_0} - \langle H_0\rangle^2_{\gamma H_0}}, \quad (4.15)$$

where

$$\gamma - 1 = (\langle\Delta H_0\rangle_{\gamma H_0} - \langle\Delta\rangle_{\gamma H_0}\langle H_0\rangle_{\gamma H_0})/(\langle H_0^2\rangle_{\gamma H_0} - \langle H_0\rangle^2_{\gamma H_0}).$$

If equation (4.15) is then carried through second order (most simply, before minimization) we obtain

$$F(H) \leq F(H_0) + \bar{\Delta} - \frac{\beta}{2}\langle(\Delta - \bar{\Delta})^2\rangle\frac{\langle(\Delta - \bar{\Delta})(H_0 - \bar{H}_0)\rangle^2}{\langle(\Delta - \bar{\Delta})^2\rangle\langle(H_0 - \bar{H}_0)^2\rangle} + \cdots \quad (4.16)$$

with all expectations at H_0. Equation (4.16) coincides with (4.13) where H_0 and Δ are linearly correlated, i.e., if Δ is interpolated as a linear function of H_0.

Proceeding to lower bounds, we have seen that the problem divides itself into two parts, first of estimating the entropy associated with a given $f^{(2)}$, and then that of not badly underestimating resulting free energy. As companionpiece to (4.13), we note, using

$$S = \beta^2 \frac{\partial F}{\partial \beta}$$

and

$$\frac{\partial}{\partial \beta}\langle A\rangle = -\langle(H_0 - \bar{H}_0)(A - \bar{A})\rangle$$

that

$$S(H) - S(H_0) = -\beta^2\langle(H_0 - \bar{H}_0)(\Delta - \bar{\Delta})\rangle - \frac{\beta^2}{2}\langle(\Delta - \bar{\Delta})^2\rangle + \frac{\beta^3}{2}\langle(H_0 - \bar{H}_0)(\Delta - \bar{\Delta})^2\rangle + \cdots, \quad (4.17)$$

while from

$$f^{(2)} = \frac{2}{N}\frac{\delta F}{\delta \Delta^{(2)}},$$

applying

$$\operatorname{Tr}(H_0^{(2)} - \bar{H}_0^{(2)})\frac{\delta}{\delta\Delta^{(2)}},$$

$$\frac{N}{2}\operatorname{Tr} H_0^{(2)}(f^{(2)} - f_0^{(2)}) = -\beta\langle(H_0 - \bar{H}_0)(\Delta - \bar{\Delta})\rangle + \frac{\beta^2}{2}\langle(H_0 - \bar{H}_0)(\Delta - \bar{\Delta})^2\rangle. \quad (4.18)$$

To first order,

$$f^{(2)}(1', 2') = f_0^{(2)}(1', 2') - \beta\langle(\Delta - \bar{\Delta})\delta(1 - 1')\delta(2 - 2')\rangle, \quad (4.19)$$

allowing $\Delta - \bar{\Delta}$ to be solved, and then

$$\frac{1}{\beta}s(H) = \frac{1}{\beta}s(H_0) + \tfrac{1}{2}\operatorname{Tr} H_0^{(2)}(f^{(2)} - f_0^{(2)}) - \frac{\beta}{2N}\langle(\Delta - \bar{\Delta})^2\rangle + \cdots. \quad (4.20)$$

The single-model condition (3.5) reads in the present case

$$\frac{1}{\beta}s(H) \leq \tfrac{1}{2}\operatorname{Tr}(H_0^{(2)} - \bar{H}_0^{(2)})f^{(2)} + \frac{1}{\beta}s(H_0), \quad (4.21)$$

again accurate to first order, thus placing the onus of the approximation to this order totally upon the further minimization over $f^{(2)}$. Accuracy to second order is achieved only if Δ turns out to be a constant.

Let us see what happens when the entropy and $f^{(2)}$ minimizing approximations are combined. Now for a single model, Eq. (3.9) [or the weaker (3.10)] is valid, yielding in the present case

$$F(H) \geq \min\Delta + F(H_0) \geq \frac{N}{2}\min\Delta^{(2)} + F(H_0), \quad (4.22)$$

which is useless unless Δ is nonnegative, and then quite poor. The corresponding $f^{(2)}$ is composed of δ functions at the minimum of Δ, concomitant with the triviality of the classical ground state, and the entropy is the crude first order

$$\frac{S}{\beta} = \frac{N}{2}\operatorname{Tr} H_0^{(2)}f^{(2)} - F(H_0), \quad (4.23)$$

illustrating the drawbacks associated with the advantage of not requiring $f_0^{(2)}$ for our lower-bound formulation.

For a discrete (i.e., spin) classical space, the δ function nature of $f^{(2)}$ is not onerous. Let us see what improvement in (4.22) and (4.23) occurs when the model set γH_0 is employed. Equation (3.7) now demands the maximum of $F(\gamma H_0) - \frac{1}{2}N\gamma \operatorname{Tr} H_0^{(2)} f^{(2)}$; this occurs at $\langle H_0\rangle_{\gamma H_0} = \frac{1}{2}N \operatorname{Tr} H_0^{(2)} f^{(2)}$, hence yielding

$$\frac{S}{\beta} = \gamma\langle H_0\rangle_{\gamma H_0} - F(\gamma H_0), \tag{4.24}$$

where

$$\langle H_0\rangle_{\gamma H_0} = \tfrac{1}{2}N \operatorname{Tr} H_0^{(2)} f^{(2)},$$

a rather obvious type of self-consistent γH_0 to produce $f^{(2)}$.

5. SCALED MODEL RESTRICTIONS FOR THE ONE-DIMENSIONAL ISING MODEL WITH EXTERNAL FIELD

We again consider in detail an illustrative example, now of lower bounds for statistical states. It is that of the paramagnetic ($\epsilon > 0$) one-dimensional Ising model with external field B, and periodic boundary conditions;

$$H = -\epsilon \sum_1^N s_k s_{k+1} - B \sum_1^N s_k, \tag{5.1}$$

where $s_k = \pm 1$ and $s_{N+1} = s_1$. This is equivalent to a lattice gas in a grand ensemble with density unequal to one half maximum. Insofar as the energy is concerned, one needs only the following:

$$f_s = \langle \delta_{s_k,s}\rangle,$$
$$f_{ss'} = \langle \delta_{s_k,s}\delta_{s_{k+1},s'}\rangle, \tag{5.2}$$

the nearest-neighbor distributions—a 2 vector and a 2×2 matrix, respectively. Indeed, since the only independent functions of two spins s, s' are $1, s, s', ss'$, the moments

$$\rho_1 = \langle \textstyle\sum s_k\rangle = N\langle s_k\rangle \equiv Nf_1,$$
$$\rho_2 = \langle \textstyle\sum s_k s_{k+1}\rangle = N\langle s_k s_{k+1}\rangle \equiv Nf_2 \tag{5.3}$$

suffice. If we stick to models which demand no more than the simple (5.2) or (5.3), i.e., one-dimensional nearest-neighbor forces, we are in fact restricted to model Hamiltonians of the form (5.1) but with different parameters, ϵ, B—a practical procedure only under special conditions.

Before concentrating on the equilibrium free energy of (5.1) and comparing with approximations, let us examine the functional dependence of S on ρ_1 and ρ_2—the crux of our approximations—for this too is exactly obtainable in our case. A convenient way to obtain this dependence in the thermodynamic limit $N \to \infty$ is to observe that since βF is a functional only of βH, then from (5.1),

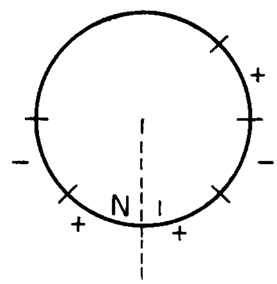


FIG. 2. Ising model configuration with periodic boundary.

$$S = \beta^2 \frac{\partial F}{\partial \beta} = \beta \frac{\partial}{\partial \beta}\beta F - \beta F$$
$$= -\beta F + \epsilon \frac{\partial}{\partial \epsilon}\beta F + B\frac{\partial}{\partial B}\beta F.$$

Hence,

$$-\frac{S}{\beta} = F - \epsilon\frac{\partial F}{\partial \epsilon} - B\frac{\partial F}{\partial B}. \tag{5.4}$$

But also from (5.1) and (5.3),

$$\rho_2 = -\frac{\partial F}{\partial \epsilon}, \quad \rho_1 = -\frac{\partial F}{\partial B}. \tag{5.5}$$

Thus, $-s/\beta$ as a function of ρ_1, ρ_2 is the Legendre transform of F as a function of ϵ and B, and in the thermodynamic limit is identical with the ρ_1, ρ_2 ensemble potential;

$$S = \ln \sum_{\{s_k\}} \delta(\textstyle\sum s_k, \rho_1)\, \delta(\sum s_k s_{k+1}, \rho_2), \tag{5.6}$$

all at fixed N. The free energy may of course be recovered as

$$F(\epsilon, B) = -\frac{1}{\beta}\ln \sum_{\rho_1,\rho_2} \exp\,[S(\rho_1,\rho_2) + \beta\rho_1 B + \beta\rho_2\epsilon]. \tag{5.7}$$

Let us evaluate (5.6). Denoting the number of spins up and down by $a_\pm$, the number of spin flips from -1 to 1 in the sequence $\{s_k\}$ by b ($=$ flip number from 1 to -1), we have

$$a_+ = \tfrac{1}{2}N(1 + f_1), \qquad a_- = \tfrac{1}{2}N(1 - f_1),$$
$$b = \tfrac{1}{4}N(1 - f_2). \tag{5.8}$$

(See Fig. 2.)

But the number of configurations at fixed a_+, a_-, b is the product of the number of ways that the b + 1's which occur immediately after a flip can be chosen from the a_+ + 1's present, by the corresponding number for -1's. In other words,

$$e^S = \binom{a_+}{b}\binom{a_-}{b}, \tag{5.9}$$

and as $N \to \infty$, we obtain from (5.8)

$$s = \frac{S}{N} = \ln 2 + \tfrac{1}{2}[(1+f_1)\ln(1+f_1) + (1-f_1)\ln(1-f_1) - (1-f_2)\ln(1-f_2)] - \tfrac{1}{4}[(f_2+2f_1+1)\ln(f_2+2f_1+1) + (f_2-2f_1+1)\ln[f_2-2f_1+1)], \quad (5.10)$$

to which any approximation must be compared.

Now the form in which we pose our test problem is this: Suppose that the field-free Ising model has been solved and we want to find the initial as well as finite effect of an added field. We thus have available the model set

$$H_M = -M\sum_1^N s_k s_{k+1} \quad (5.11)$$

on which to base a restricted variational principle. Clearly,

$$\sum e^{-\beta H_M} = \sum \prod e^{\beta M s_k s_{k+1}} = \mathrm{Tr}\begin{pmatrix} e^{\beta M} & e^{-\beta M} \\ e^{-\beta M} & e^{\beta M}\end{pmatrix}^N,$$

and the maximum eigenvalue of the 2×2 matrix is $\lambda_{\max} = 2\cosh\beta M$, so that

$$F(M,0) = -\frac{N}{\beta}\ln(2\cosh\beta M). \quad (5.12)$$

Our aim is to reproduce $F(H)$, which can be evaluated in precisely the same way, yielding

$$F(\epsilon, B) = -\frac{N}{\beta}\ln\left(e^{\beta\epsilon}\cosh\beta B + (e^{-2\beta\epsilon} + e^{2\beta\epsilon}\sinh^2\beta B)^{\frac{1}{2}}\right) = -N\epsilon - \frac{N}{\beta}\ln\left(\cosh\beta B + (e^{-4\beta\epsilon} + \sinh^2\beta B)^{\frac{1}{2}}\right). \quad (5.13)$$

We first consider the very weakest forms Eqs. (3.9) and (3.10), which become

$$F(H) \geq \max_M\left(E_0(\epsilon - M, B) + F(M, 0)\right) \geq \max_M\left(\frac{N}{2}E_0^{(2)}(\epsilon - M, B) + F(M,0)\right). \quad (5.14)$$

Now by checking each of the four cases, we see that $-\epsilon ss' - \frac{1}{2}B(s+s')$ has a minimum of

$$-\left|\epsilon + \tfrac{1}{2}|B|\right| - \tfrac{1}{2}|B|.$$

It follows that

$$E_0(\epsilon, B) = \frac{N}{2}E_0^{(2)}(\epsilon, B) = -N\left|\epsilon + \tfrac{1}{2}|B|\right| - \tfrac{1}{2}N|B|, \quad (5.15)$$

Eq. (3.10), here, being no weaker than Eq. (3.9); we have

$$F(H) \geq -N\min_M\left(\frac{1}{\beta}\ln 2\cosh\beta M + \left|\epsilon + \tfrac{1}{2}|B| - M\right|\right) - \tfrac{1}{2}N|B|, \quad (5.16)$$

yielding on evaluation

$$F(H) \geq -N\epsilon - \frac{N}{\beta}\ln(e^{\beta|B|} + e^{-2\beta\epsilon}) \quad (5.17)$$

at the optimum model $M = \epsilon + \frac{1}{2}|B|$. For the corresponding entropy approximation, (3.6) now reads

$$\frac{s}{\beta} \leq -\max_M\left[(1/N)F(M,0) + Mf_2\right], \quad (5.18)$$

yielding on evaluation

$$s \leq \ln 2 - \tfrac{1}{2}[(1+f_2)\ln(1+f_2) + (1-f_2)\ln(1-f_2)]. \quad (5.19)$$

Let us proceed without a pause to the stronger principle (3.7), or in the present case,

$$F(H) \geq N\min_{f_1,f_2}\max_M\left[(M-\epsilon)f_2 - Bf_1 - \frac{1}{\beta}\ln(2\cosh\beta M)\right], \quad (5.20)$$

essentially a transcription of (5.18). Thus,

$$F(H) \geq \frac{N}{\beta}\min_{f_1,f_2}\left[-\ln 2 + \tfrac{1}{2}(1+f_2)\ln(1+f_2) + \tfrac{1}{2}(1-f_2)\ln(1-f_2) - \beta\epsilon f_2 - \beta Bf_1\right]. \quad (5.21)$$

The minimization of (5.21) now does depend on the allowed domain of f_1 and f_2. Pure two-body restrictions result from the positivity and normalization of $f_{ss'}$ of (5.2):

$$f_{++} \geq 0,\quad f_{--} \geq 0,\quad f_{+-} = f_{-+} \geq 0,$$
$$f_{++} + f_{--} + f_{+-} + f_{-+} = 1. \quad (5.22)$$

Thus,

$$f_1 = f_{++} - f_{--},$$
$$f_2 = 2(f_{++} + f_{--}) - 1 \quad (5.23)$$

must satisfy the conditions

$$|f_1| \leq \tfrac{1}{2}(1+f_2) \leq 1, \quad (5.24)$$

which applied to (5.21), result in

$$F(H) \geq F = \frac{N}{\beta}\min_{f_2}\left[-\ln 2 + \tfrac{1}{2}(1+f_2)\ln(1+f_2) + \tfrac{1}{2}(1-f_2)\ln(1-f_2) - \beta\epsilon f_2 - \tfrac{1}{2}\beta|B|(1+f_2)\right] = N\left(-\epsilon - \frac{1}{\beta}\ln(e^{\beta|B|} + e^{-2\beta\epsilon})\right)$$

at

$$f_2 = \tanh \beta(\epsilon + \tfrac{1}{2}|B|),$$
$$f_1 = \tfrac{1}{2}(1 + f_2)\,\mathrm{sgn}\, B$$
$$= \tfrac{1}{2} \exp [\beta(\epsilon + \tfrac{1}{2}|B|) \,\mathrm{sech}\, (\epsilon + \tfrac{1}{2}|B|), \quad (5.25)$$

the lower bound F $(= F(\epsilon + \frac{1}{2}|B|, 0) - (N/2)|B|)$ being precisely as in (5.17); the unrestricted (5.17) is right on the boundary of the restriction (5.24). We may also compare with the correct f_1 and f_2, determined by (5.5) and (5.13):

$$f_1 = -\frac{1}{N}\frac{\partial F}{\partial B} = (e^{-4\beta\epsilon} + \sinh^2 \beta B)^{-\frac{1}{2}} \sinh \beta B,$$

$$f_2 = -\frac{1}{N}\frac{\partial F}{\partial \epsilon} = \coth 2\beta\epsilon - (e^{-4\beta\epsilon} + \sinh^2 \beta B)^{-\frac{1}{2}}$$
$$\cdot \cosh \beta B e^{-2\beta\epsilon} \,\mathrm{csch}\, 2\beta\epsilon. \quad (5.26)$$

Finally, we ask how (5.24) is altered by a full N-body restriction. We want to replace (5.24) by the allowed domain of the full N-body reduced f_2, given f_1. Clearly the stationary values of f_2, given f_1, are obtained from those of $f_2 + \lambda f_1$, corresponding to the $\beta \to \infty$ limit of the statistical problem for

$$H = \pm(\sum s_k s_{k+1} + \lambda \sum s_k).$$

But the solution of this problem is (5.26). Solving (5.26), we have

$$\tanh \beta B = f_1[(1 - f_1^2)e^{4\beta\epsilon} + f_1^2]^{-\frac{1}{2}}, \quad (5.27)$$

so that, setting $X = e^{4\beta\epsilon} - 1 \geq -1$,

$$f_2 = 1 + \frac{2}{X} - \frac{2}{X}[1 + (1 - f_1^2)X]^{\frac{1}{2}}. \quad (5.28)$$

The domain of f_2 given by (5.28) is (by virtue of $|f_1| \leq 1$) again

$$-1 + 2|f_1| \leq f_2 \leq 1, \quad (5.29)$$

identical with (5.24). Hence, the result (5.17) is not altered on applying the full strength of N-body realizability. The complete approximation in this example is that of too weakly bounding the entropy. This is neither surprising nor especially deplorable, since an expansion of Eq. (5.10) about the model condition $f_1 = 0$ yields

$$s = \ln 2 - \tfrac{1}{2}[(1 + f_2)\ln(1 + f_2)$$
$$+ (1 - f_2)\ln(1 - f_2)] - \frac{1}{2}\frac{1 - f_2}{1 + f_2} f_1^2 \cdots, \quad (5.30)$$

a second-order correction to (5.19) (and since $1 - f_2/1 + f_2 = e^{-2\beta\epsilon}$ for the model, particularly small at low temperature).

Matrix Products and the Explicit 3, 6, 9, and 12-j Coefficients of the Regular Representation of $SU(n)$†

L. M. KAPLAN‡ AND M. RESNIKOFF
Department of Physics and Astronomy, University of Maryland, College Park, Maryland

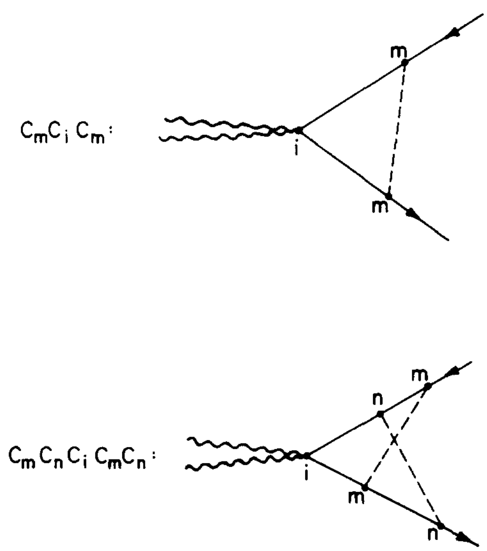
(Received 4 August 1966)

The explicit Wigner coefficients are determined for the direct product of regular representations, $(N) \otimes (N) = 2(N) + \cdots$, of $SU(n)$, where $N = n^2 - 1$. Triple products $C_m C_i C_m = \alpha F_i + \beta D_i$, and higher-order products, are calculated, where C_i may be F_i or D_i, the $N \times N$ Hermitian matrices of the regular representation, and m is summed. The coefficients α, β are shown to be 6-j symbols, and higher-order products yield the explicit 9-j, 12-j, symbols. A theorem concerning (3p)-j coefficients is proved.

I. INTRODUCTION

THE ostensible purpose of this paper is to explicitly determine the 3, 6, 9, 12-j coefficients of $SU(n)$ for the special case when the regular representation occurs in the direct product of regular representations, $(N) \otimes (N) = 2(N) + \cdots$, where $N = n^2 - 1$. This, it must be admitted, was not the initial motivation for the work.

The *real* purpose was to consider products of the 8×8 Hermitian matrices C_k, $C_m C_i C_m$ or $C_m C_n C_i C_m C_n$, where m, n are summed from 1 to 8, and C_k is F_k or D_k, the Gell-Mann matrices.[1] In a special current algebra model,[2] $(C_i)_{jk}$ represented the unrenormalized lepton-current-baryon vertex, and the various 3rd, 5th, and higher-order products represented exchange meson corrections to this vertex. As an example, the above two products are represented graphically in Fig. 1. The question posed was the following. Say the original vertex is (D) F-type. Under what conditions would the various corrections leave an (D) F-type vertex? In general, then, $C_m C_i C_m = \alpha F_i + \beta D_i$, and the original question reduced to the calculation of α, β. However, it became clear, quite shortly, that α and β were 6-j coefficients which relate the manner in which three octet representations may be coupled to yield another octet representation. Higher-order products provided the 9, 12-j coefficients. The methods derived to evaluate such products were quite general and could, indeed, be applied to matrices of the regular representation of $SU(n)$. This, then, provided a way to evaluate the 6, 9, 12-j coefficients for $SU(n)$, the results of this paper.



THE INCOMING, OUTGOING LINES ARE BARYONS, THE WAVY LINES, A LEPTON CURRENT, AND THE DASHED LINES, EXCHANGED VIRTUAL MESONS.

FIG. 1. Vertex corrections.

This paper exploits an intimate relation between the fundamental and regular representations of the Lie algebra, namely that the $n \times n$ matrices serve as the carrier space of the regular representation; in fact, define the coupling coefficients. This relation has been employed by Gell-Mann[1] and Lee[3] for $SU(3)$ and noted by Bargmann,[4] Behrends *et al.*,[5] and Biedenharn,[6,7] among others. In Sec. II, the $n \times n$ Hermitian matrices λ_i, $i = 1, \cdots, N$, the matrices of the

† Supported by the U.S. Air Force under contract AFOSR 500-66.
‡ Present address: Congregation B'nai Sholom, Blountville, Tenn.
[1] M. Gell-Mann, California Institute of Technology, Pasedena, California, Report CTSL-20 (1961).
[2] This line of approach was suggested by Professor J. Sucher and will be reported elsewhere.
[3] B. W. Lee, International Center for Theoretical Physics Report (1965).
[4] V. Bargmann, Ann. Math. **48**, 568 (1947).
[5] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962).
[6] L. C. Biedenharn, J. Math. Phys. **4**, 436 (1963).
[7] G. E. Baird and L. C. Biedenharn, J. Math. Phys. **6**, 1847 (1965).

infinitesimal generators of $SU(n)$, are defined, and the Gell-Mann matrices F_i, D_i, constructed. It is then possible, by means of transformation vectors $\epsilon_i^{(\mu)}$, to transform from the λ_i to a non-Hermitian "spherical" basis $L^{(\mu)}$, $\mu = 1, \cdots, N$, and define the states μ in terms of the weights. This transformation allows us to express the Wigner coefficients in terms of linear combinations of the F_i, D_i, matrices, as shown in Sec. III. The appropriate phase convention for the Wigner coefficients is introduced through the spherical vectors. The explicit Wigner coefficients then have the conventional orthogonality, symmetry, properties, and are equal to de Swart's[8] for the case of $SU(3)$. In Sec. IV, the products of matrices $C_m C_i C_m = \alpha F_i + \beta D_i$ are considered, and the α, β coefficients are shown to be 6-j symbols. Higher-order products are 9, 12-j symbols. A general theorem concerning products of C_k is proven here: if the product contains an odd number of F's, then $\beta = 0$; if the product contains an odd number of D's, then $\alpha = 0$. The explicit calculation of the coefficients α, β for the various products is relegated to Appendix A where the calculation is self-contained (and simply connected). The results are listed in three tables. The matter of phase convention for the Wigner coefficients is discussed in Appendix B.

II. INFINITESIMAL MATRICES, WEIGHTS

A. Hermitian Basis

Let the $n \times n$ matrices λ_i, $i = 1, \cdots, N = n^2 - 1$, be the matrices of the infinitesimal generators of $SU(n)$. The matrices λ_i are Hermitian and traceless. Since the matrices λ_i and 1 span the space of $n \times n$ Hermitian matrices, the product $\lambda_i\lambda_j$ may be expressed as

$$\lambda_i\lambda_j = a\delta_{ij}1 + c_{ijk}\lambda_k, \tag{1}$$

where $c_{ijk} = d_{ijk} + if_{ijk}$, and f, d are real. If we combine Eq. (1) and the Hermitian conjugate of Eq. (1), assuming λ_i constructed such that a is real, we have the relations

$$[\lambda_i, \lambda_j] = 2if_{ijk}\lambda_k, \tag{2a}$$

$$\{\lambda_i, \lambda_j\} = 2a\delta_{ij}1 + 2d_{ijk}\lambda_k, \tag{2b}$$

$$\mathrm{Tr}\,(\lambda_i\lambda_j) = na\delta_{ij}, \tag{2c}$$

where [ ] is the commutator, { } is the anticommutator. Equation (2a) shows that the λ_i are a basis of the Lie algebra. The matrices λ_i may be normalized so that a, in Eq. (2c), is independent of the indices i, j and $a = 2/n$.[9] Then,

$$\mathrm{Tr}\,(\lambda_i\lambda_j) = 2\delta_{ij}. \tag{2c$'$}$$

In Appendix A, however, a will be left arbitrary since the specific realization is not important for the calculation of matrix products. From Eqs. (2a,b), $f_{ijk}(d_{ijk})$ is antisymmetric (symmetric) under the interchange i, j. However, using Eq. (2c), we have that

$$\mathrm{Tr}\,([\lambda_i, \lambda_j]\lambda_k) = (2an)if_{ijk}, \tag{3a}$$

$$\mathrm{Tr}\,(\{\lambda_i, \lambda_j\}\lambda_k) = (2an)\,d_{ijk}. \tag{3b}$$

From these equations and the fact that λ_i is Hermitian, it may be shown that $f_{ijk}(d_{ijk})$ is antisymmetric (symmetric) under the exchange of any two indices. Given the specific basis λ_i, the coefficients f, d, are completely determined by Eqs. (3).

In analogy to the $SU(3)$ case,[1] let the Gell-Mann matrices F_i, D_i, $i = 1, \cdots, N$, be defined

$$(F_i)_{jk} = -if_{ijk}, \quad (D_i)_{jk} = d_{ijk}. \tag{4}$$

Due to the symmetry of the f, d coefficients, and the fact that F_i is pure imaginary, D_i real, the matrices F_i, D_i are Hermitian. The matrices are also traceless (see Appendix A).

If a set of operators $\{T_j\}$ satisfy the commutation relations

$$[F_i, T_j] = if_{ijk}T_k, \tag{5}$$

the set $\{T_j\}$ are vector operators of $SU(n)$.[6,10] From the two Jacobi identities, Eqs. (A1), (A3) of the Appendix, the relations follow

$$[F_i, F_j] = if_{ijk}F_k, \tag{6a}$$

$$[F_i, D_j] = if_{ijk}D_k. \tag{6b}$$

Hence, the Gell-Mann matrices, Eq. (4), are vectors of $SU(n)$.

The coefficients f_{ijk}, d_{ijk} also serve as the coupling coefficients of $SU(n)$.[11] Say that two sets of vectors, $(T_k^{(1)})$, $(T_k^{(2)})$, satisfy Eq. (5). The direct product set $\{T_k^{(1)}T_m^{(2)}\}$ may be reduced with the coefficients f, d,

$$T_j^{(3)} = b_1 f_{jkm}T_k^{(1)}T_m^{(2)}, \quad T_j^{(4)} = b_2\, d_{jkm}T_k^{(1)}T_m^{(2)}, \tag{7}$$

where the constants b_i are independent of jkm. The sets, $\{T_j^{(3)}\}$, $\{T_j^{(4)}\}$, are easily shown to satisfy Eq. (5) [using Eqs. (6)], and, hence, also constitute vector operators.

The set of coupling coefficients f_{ijk}, d_{ijk}, satisfy the

[8] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

[9] M. Gell-Mann (Ref. 1) chooses $a = \frac{2}{3}$ for $SU(3)$, as a generalization of the normalization for the Pauli spin matrices, $\mathrm{Tr}\,(\tau_i\tau_j) = 2\delta_{ij}$.

[10] A relation similar to this is derived by A. P. Stone, Proc. Cambridge Phil. Soc. **57**, 460 (1961) for the general semi-simple Lie group.

[11] This is shown by Biedenharn (Ref. 6) for a specific realization of the regular representation. The equivalent result, that the coefficients d_{ijk} may be used to construct the Casimir invariants of $SU(n)$, was proved by A. Klein, J. Math. Phys. **4**, 1283 (1963).

orthogonality relations,

$$\sum_{ij} f_{ijk} f_{ijk'} = \mathrm{Tr}\,(F_k F_{k'}) = a_f \delta_{kk'} \tag{8a}$$

$$\sum_{ij} d_{ijk}\, d_{ijk'} = \mathrm{Tr}\,(D_k D_{k'}) = a_d \delta_{kk'} \tag{8b}$$

$$\sum_{ij} f_{ijk}\, d_{ijk'} = -i\,\mathrm{Tr}\,(F_k D_{k'}) = 0, \tag{8c}$$

as shown in Appendix A [see Eq. (A11)], where $a_f = \frac{1}{2}a(N+1)$, $a_d = \frac{1}{2}a(N-3)$.

A specific realization of the $n \times n$ matrices λ_i, $i = 1, \cdots, N$, may be given, in analogy to the $SU(2)$ Pauli spin matrices, and the $SU(3)$ matrices.[1] Let $e_{(ij)}$ be the matrix with 1 in the ijth position, and zero elsewhere. Then,

$$(e_{(ij)})_{uv} = \delta_{iu}\,\delta_{jv}. \tag{9}$$

Let $\alpha \equiv (ij)$ be the ordered pairs $i < j$, where $i, j = 1, \cdots, n$, and let $-\alpha \equiv (ji)$. Then, for the nondiagonal λ_i matrices,[12] we define

$$\lambda_\alpha^{(1)} = e_\alpha + e_{-\alpha} = e_{(ij)} + e_{(ji)}, \tag{10a}$$

$$\lambda_\alpha^{(2)} = -i(e_\alpha - e_{-\alpha}) = -i(e_{(ij)} - e_{(ji)}). \tag{10b}$$

The matrices $\lambda_\alpha^{(1)}$, $\lambda_\alpha^{(2)}$, $\alpha = 1, \cdots, \frac{1}{2}m$, $[m = n(n-1)]$, satisfy Eq. (2c′). For the diagonal matrices,[13] let

$$(\hat{h}_k)_{mn} = d_m^{(k)} \delta_{mn} a_k \qquad \begin{matrix} m, n = 1, \cdots, n \\ k = 1, \cdots, n-1, \end{matrix} \tag{11}$$

where

$$\begin{aligned} d_m^{(k)} &= 1, & m &= 1, \cdots, k; \\ &= -k, & m &= k+1; \\ &= 0, & m &= k+2, \cdots, n. \end{aligned}$$

This set of diagonal matrices[14] $\lambda_{m+k} = \hat{h}_k$, satisfies the orthonormality conditions, Eq. (2c′), if the normalization constant is taken

$$a_k = \sqrt{2}[k(k+1)]^{-1/2}. \tag{12}$$

The explicit set of Hermitian matrices[15] λ_i, $i = 1, \cdots, N$, and Eqs. (10) and (11), uniquely define the matrices F_i, D_i, by Eq. (3).

B. Spherical Basis

The conventional Wigner coefficients are defined in terms of a non-Hermitian "spherical" basis. In $SU(2)$, the Hermitian set I_i, $i = 1, 2, 3$, are vectors of the regular representation satisfying the commutation relations, $[I_i, I_j] = i\epsilon_{ijk} I_k$, similar to Eq. (6a). The spherical set [with phase $(-1)^m$] is $I_+ = (-1/\sqrt{2})(I_1 + iI_2)$, $I_- = (1/\sqrt{2})(I_1 + iI_2)$, $I_0 = I_3$. The set of operators I_+, I_0, I_-, may also be associated with the set of states, $m = +1, -0, -1$, of the regular representation.[16] This transformation, from the Hermitian basis to the spherical basis, and the labeling of states of the regular representation, may be carried out for $SU(n)$, as a generalization of the $SU(2)$, $SU(3)$ cases.

Let the spherical basis $L^{(\alpha)}$, $\alpha = 1, \cdots, N$, be expressed as a linear combination of the Hermitian basis λ_i by means of the transformation vectors $\epsilon_i^{(\alpha)}$,[17]

$$L^{(\alpha)} = \epsilon_i^{(\alpha)}(-1)^{Q_\alpha}\lambda_i = \frac{(-1)^{Q_\alpha}}{\sqrt{2}}(\lambda_\alpha^{(1)} + i\lambda_\alpha^{(2)}), \tag{13a}$$

$$L^{(-\alpha)} = \epsilon_i^{(-\alpha)}\lambda_i = \frac{1}{\sqrt{2}}(\lambda_\alpha^{(1)} - i\lambda_\alpha^{(2)}), \quad \alpha = 1, \cdots, m/2, \tag{13b}$$

$$L^{(\alpha)} = \hat{h}_i \quad \alpha = m + i = m+1, \cdots, N, \tag{13c}$$

where Q_α is the generalized "charge," an integer (see Appendix B),

$$Q_\alpha = \sum_{k=1}^{n-1} h_k / k \tag{13d}$$

and h_k are the weights of the $\alpha = (ij)$ state (see Sec. IIC).[18] The transformation vectors have the built-in phase convention

$$(\epsilon_i^{(\alpha)})^* = (-1)^{Q_\alpha}\epsilon_i^{(-\alpha)}, \tag{14}$$

a generalization of the $SU(3)$ case,[3] and the orthogonality properties,

$$\sum_{i=1}^{m} \epsilon_i^{(\alpha)}\epsilon_i^{(\beta)*} = \delta_{\alpha\beta}, \qquad \sum_{\alpha=-m/2}^{m/2} \epsilon_i^{(\alpha)}\epsilon_j^{(\alpha)*} = \delta_{ij}, \tag{15a,b}$$

$$\sum_{i=1}^{m} \epsilon_i^{(\alpha)}\epsilon_i^{(\beta)} = \delta_{\alpha,-\beta}(-1)^{Q_\alpha}, \qquad \sum_{\alpha=-m/2}^{m/2} \epsilon_i^{(\alpha)}\epsilon_j^{(\alpha)} = \delta_{ij}\phi_i, \tag{15c,d}$$

where

$$\phi_i = \begin{cases} -1 & i = 1, \cdots, m/2 \text{ associated with } \lambda_i^{(2)}, \\ +1 & i = 1, \cdots, m/2 \text{ associated with } \lambda_i^{(1)}. \end{cases}$$

---

[12] C. Itzykson and M. Nauenberg, Rev. Mod. Phys. **38**, 95 (1966).

[13] These matrices were used by G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963), to show that it is possible to find commuting subgroups $U(1)$, $SU(n-1)$, of $SU(n)$.

[14] The matrices (or operators) are denoted by $\hat{h}_k$, and the eigenvalues, h_k.

[15] Apart from normalization and phase, this is the same set of matrices employed by A. Pais, Rev. Mod. Phys. **38**, 215 (1966).

[16] The association of operators and states for $SU(n)$ has been extensively discussed by Biedenharn (Ref. 6), and G. E. Baird and L. C. Biedenharn (Ref. 13) and J. Math. Phys. **5**, 1723, 1730 (1964).

[17] The transformation from a Hermitian to a spherical $SU(3)$ basis is given by B. W. Lee (Ref. 3).

[18] As shown in Appendix B, Eq. (13d) may be employed to define a consistent phase convention for the $SU(n)$ Wigner coefficients, as a generalization of the convention of de Swart (Ref. 8) and L. C. Biedenharn, Phys. Letters **3**, 69 (1962) [but not Biedenharn (Ref. 16)].

The spherical basis $L^{(\mu)}$ may be expressed in terms of the matrices $e_{(ij)}$, Eq. (9),

$$L^{(\alpha)} = \sqrt{2}(-1)^{Q_\alpha} e_\alpha = \sqrt{2}(-1)^{Q_{(ij)}} e_{(ij)},$$
$$L^{(-\alpha)} = \sqrt{2}\, e_{-\alpha} = \sqrt{2}\, e_{(ji)}. \tag{16}$$

In this form, the basis $L^{(\mu)}$ satisfies the commutation relations

$$[L^{(\alpha)}, L^{(\beta)}] = \sqrt{2}\, L^{(\gamma)}(-1)^\theta \quad \alpha = (kl),\quad \beta = (lm),\quad \gamma = (km), \tag{17a}$$
$$= -\sqrt{2}\, L^{(\gamma)}(-1)^\theta \quad \alpha = (kl),\quad \beta = (mk),\quad \gamma = (ml), \tag{17b}$$
$$= 0, \quad \text{otherwise} \quad (\alpha \neq -\beta),$$
$$[\hat{h}_k, L^{(\alpha)}] = a_k(d_i^{(k)} - d_j^{(k)})L^{(\alpha)} \equiv \alpha^{(k)} L^{(\alpha)}, \tag{17c}$$
$$[\hat{h}_i, \hat{h}_j] = 0, \tag{17d}$$
$$[L^{(\alpha)}, L^{(-\alpha)}] = (-1)^{Q_\alpha} \sum_k \alpha^{(k)} \hat{h}_k, \tag{17e}$$

where a_k is the normalization, Eq. (12), $d_i^{(k)}$ are the diagonal elements, Eq. (11), and θ is Q_α, Q_β, and Q_γ, for each α, β, γ positive, respectively.

The basis $L^{(\mu)}$ also satisfies the anticommutation relations

$$\{L^{(\alpha)}, L^{(\beta)}\} = \sqrt{2}\, L^{(\gamma)}(-1)^\theta, \quad \alpha = (kl),\quad \beta = (lm),\quad \gamma = (km) \tag{18a}$$
$$\alpha = (kl),\quad \beta = (mk),\quad \gamma = (ml) \tag{18b}$$
$$= 0, \quad \text{otherwise}, \quad (\alpha \neq -\beta),$$
$$\{\hat{h}_k, L^{(\alpha)}\} = a_k(d_i^{(k)} + d_j^{(k)})L^{(\alpha)} \equiv \beta^{(k)} L^{(\alpha)}, \tag{18c}$$
$$\{\hat{h}_i, \hat{h}_i\} = 4/n \cdot 1 + 2 \sum_k d_i^{(k-1)} \hat{h}_k a_k, \tag{18d}$$
$$\{\hat{h}_i, \hat{h}_j\} = 2a_j \hat{h}_i \quad i < j, \tag{18e}$$
$$\{L^{(\alpha)}, L^{(-\alpha)}\} = (4/n)(-1)^{Q_\alpha} \cdot 1 + (-1)^{Q_\alpha} \sum_k \beta^{(k)} \hat{h}_k. \tag{18f}$$

C. Weights

The group $SU(n)$ may be decomposed by the chain $SU(n) \supset U(1) \otimes SU(n-1)$. If we renormalize the diagonal operators

$$\hat{h}_k = \frac{1}{k}\{e_{(11)} + \cdots + e_{(kk)} - k e_{(k+1k+1)}\}, \tag{19a}$$

the decomposition may be written in terms of the eigenvalues h_{n-1}:

$$SU(n)\ \text{F.R.}: \begin{cases} h_{n-1} = 1/n, \quad SU(n-1)\ \text{F.R.} \\ h_{n-1} = 1/n - 1 = -(n-1)/n, \\ \qquad\qquad SU(n-1)\ \text{singlet}, \end{cases} \tag{19b}$$

where F.R. is the fundamental representation. The $SU(n-1)$ F.R. may be similarly decomposed. This is a decomposition in the weights of the subgroup chain[6] $SU(n) \supset SU(n-1) \supset \cdots \supset SU(2)$; the F.R. of $SU(n)$ contains the F.R. of $SU(n-1)$, and so on.

The direct product $(n) \otimes (n)^*$, where * denotes the conjugate representation, may be reduced by simple Young tableau techniques,[19]

$$(n) \otimes (n)^* = (n^2 - 1) + (1). \tag{20a}$$

In terms of the subgroup decomposition, the $SU(n)$ regular representation (written R.R.) in terms of the weights h_{n-1} and the subgroup $SU(n-1)$, is

$$SU(n)\ \text{R.R.}: \begin{cases} h_{n-1} = 1,\ SU(n-1)\ \text{F.R.} \\ h_{n-1} = 0,\ SU(n-1)\ \text{R.R.} \\ \qquad\qquad + SU(n-1)\ \text{singlet} \\ h_{n-1} = -1,\ SU(n-1)\ \text{conjugate F.R.} \end{cases} \tag{20b}$$

The R.R. of $SU(n-1)$, $SU(n-2), \cdots, SU(3)$, may be similarly decomposed.

The states of the $SU(n)$ R.R. may be associated with the matrices $e_{(ij)}$ in a manner similar to the association of the $SU(2)$, $m = +1, -1, 0$ states with the matrices $e_{(12)}$, $e_{(21)}$, $e_{(11)}$, $-e_{(22)}$. The matrices $e_{(in)}$, $i = 1, \cdots, n-1$, associated with the states $h_{n-1} = 1$, $SU(n-1)$ F.R., are given in Table I. The matrices $e_{(ni)}$, associated with the states $h_{n-1} = -1$, $SU(n-1)$ conjugate F.R., follow similarly; the weights are the negative of those in Table I. The matrices $e_{(ij)}$, $i, j = 1, \cdots, n-1$, are associated with the $h_{n-1} = 0$, $SU(n-1)$ R.R. The matrix $e_{(nn)}$, or the traceless $\hat{h}_{n-1}$, is associated with the state $h_{n-1} = 0$, $SU(n-1)$ singlet. There are, in all, $n-1$ singlet states associated with the center of the Cartan algebra.[20] This decomposition procedure may now be repeated for the $SU(n-1)$ regular representation. For example, the matrices $e_{(in-1)}$, $i = 1, \cdots, n-2$, are associated with the states $h_{n-2} = 1$, $SU(n-2)$ F.R., and are given by a table similar to Table I. This reduction of the $SU(n)$ R.R. is simply an application of the Weyl branching law[21] to $SU(n)$.

III. WIGNER COEFFICIENTS

A. Definition

Employing the transformation vectors $\epsilon_i^{(\alpha)}$, Eq. (1) may be transformed to the spherical basis $L^{(\mu)}$.

[19] H. Harari, J. Math. Phys. 7, 283 (1966).
[20] G. Racah, Princeton lecture notes (1951).
[21] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., 1932), p. 390.

The commutation, anticommutation relations become

$$[L^{(\alpha)}, L^{(\beta)}] = 2bf(\alpha, \beta, \gamma)L^{(\gamma)}, \tag{21a}$$

$$\{L^{(\alpha)}, L^{(\beta)}\} = 2a\delta_{\alpha,-\beta}(-1)^{Q_\alpha}1 + 2cd(\alpha, \beta, \gamma)L^{(\gamma)}, \tag{21b}$$

where

$$bf(\alpha, \beta, \gamma) = if_{ijk}\epsilon_i^{(\alpha)}\epsilon_j^{(\beta)}\epsilon_k^{(\gamma)*}, \tag{22a}$$

$$cd(\alpha, \beta, \gamma) = d_{ijk}\epsilon_i^{(\alpha)}\epsilon_j^{(\beta)}\epsilon_k^{(\gamma)*}, \tag{22b}$$

and the b, c are normalization coefficients, independent of α, β, γ. From Eqs. (22), we see that the coefficients $f(\alpha, \beta, \gamma)$, $d(\alpha, \beta, \gamma)$ may be given as linear combinations of the f_{ijk}, d_{ijk}, respectively, using the transformation vectors $\epsilon_i^{(\alpha)}$. By Eqs. (21), we see that the coefficients $f(\alpha, \beta, \gamma)$, $d(\alpha, \beta, \gamma)$, are also given explicitly by Eqs. (17) and (18). Note that f, d are real.

The f, d coefficients in the spherical basis are precisely the Wigner coefficients coupling $(N) \otimes (N) = 2(N) + \cdots$. According to Stone,[10] the matrix elements of the spherical vectors, taken between states of the regular representation, may be expressed in terms of the structure constants of the algebra,

$$\langle\gamma| L^{(\alpha)} |\beta\rangle = [c^{(\beta)}/c^{(\gamma)}]c^\gamma_{\alpha\beta} \tag{23}$$

$$(\beta, \gamma \text{ not summed}),$$

where $c^\gamma_{\alpha\beta} = 2bf(\alpha, \beta, \gamma)$ are the structure constants. Equation (23) follows from the fact that the infinitesimal operators are also tensor operators. Since Hermitian, the matrix elements satisfy the relation

$$\langle\gamma| L^{(-\alpha)} |\beta\rangle(-1)^{Q_\alpha} = \langle\beta| L^{(\alpha)} |\gamma\rangle^*. \tag{24}$$

We then have the result

$$|c(\beta)|^2/|c(\gamma)|^2 = g_{\gamma,-\gamma}/g_{\beta,-\beta}(-1)^{Q_\alpha}, \tag{25}$$

where g_{uv} is the metric $g_{uv} = c^\beta_{\mu\alpha}c^\alpha_{v\beta}$. Using Eqs. (8) and (15), we may evaluate the metric

$$g_{\mu\nu} = (-1)^{Q_\mu}\delta_{\mu,-\nu}(2b)^2.$$

Since $(-1)^{Q_\alpha+Q_\beta-Q_\gamma} = +1$, from the definition Eq. (13d) and the fact that α, β, γ satisfy the triangle relations required by the commutation relations, Eq. (17), we see that $|c(\beta)|^2 = |c(\gamma)|^2$, independent of β, γ. The ratio $c^{(\beta)}/c^{(\gamma)}$ is simply a phase which may be appropriately chosen. Choose the phase such that

$$\langle\gamma| L^{(\alpha)} |\beta\rangle = (-1)^\omega 2bf(\alpha, \beta, \gamma), \tag{26a}$$

where

$$(-1)^\omega = \begin{cases} +1 & \text{if } \alpha, \beta, \gamma = \pm 1, \cdots, \pm m/2 \\ -1 & \text{if } \alpha, \beta \text{ or } \gamma = m+1, \cdots, N. \end{cases} \tag{26b}$$

This is the requirement that

$$\langle\gamma| L^{(1j)} |\beta\rangle \geq 0, \quad j = 2, \cdots, n, \tag{27}$$

an immediate generalization of the Condon–Shortley[22] phase convention for $SU(2)$ and the Biedenharn[18] phase convention for $SU(3)$. The fact that Eq. (27) follows from the definitions Eq. (26) and (13d) is shown in Appendix B.

According to the Wigner–Eckart theorem,

$$\langle\gamma| L^{(\alpha)} |\beta\rangle = \sum_{\rho=1,2} \begin{pmatrix} N & N & N \\ \alpha & \beta & \gamma \end{pmatrix}_\rho \langle\|N\|\rangle_\rho, \tag{28}$$

where the first factor in the sum is the Wigner coefficient and the second factor is the reduced matrix element, independent of the row labels α, β, γ. The conventional choice[23] is to take $\langle\|N\|\rangle_2 = 0$. Then the Wigner coefficient is given directly by the matrix element of the generators,

$$\begin{pmatrix} N & N & N \\ \alpha & \beta & \gamma \end{pmatrix}_1 \langle\|N\|\rangle_1 = 2bf(\alpha, \beta, \gamma)(-1)^\omega. \tag{29}$$

Using Eqs. (15) and (8), we have

$$\sum_{\alpha\beta} f(\alpha, \beta, \gamma)f(\alpha, \beta, \gamma') = \delta_{\gamma\gamma'}, \tag{30a}$$

$$\sum_{\alpha\beta} d(\alpha, \beta, \gamma)\, d(\alpha, \beta, \gamma') = \delta_{\gamma\gamma'}, \tag{30b}$$

$$\sum_{\alpha\beta} f(\alpha, \beta, \gamma)\, d(\alpha, \beta, \gamma') = 0, \tag{30c}$$

providing the normalizations b, c are chosen[24]

$$(2b)^2 = a_f, \quad (2c)^2 = a_d. \tag{31}$$

Since the coefficient $f(\alpha, \beta, \gamma)$ is orthogonal and normalized to unity, we see from Eq. (29) that $f(\alpha, \beta, \gamma)$ is exactly the Wigner coefficient (apart from the phase factor which is required because of our phase convention). The second set of Wigner coefficients is determined by the fact that it must be orthogonal to the first set and normalized to unity. Up to an over-all phase, this is sufficient to completely determine this coefficient.[25]

Equations (17), (18) and (21), and the phase conventions, completely determine the coefficients $f(\alpha, \beta, \gamma)$, $d(\alpha, \beta, \gamma)$. The particular association of the ordered pairs $\alpha = (ij)$ with the physical states is determined by Eq. (20c) and Table I.

---

[22] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935).

[23] This choice is made by de Swart (Ref. 8), J. G. Kuriyan, D. Lurie, and A. J. Macfarlane, J. Math. Phys. **6**, 722 (1965), and K. T. Hecht, Nucl. Phys. **62**, 1 (1965).

[24] This agrees with the factors $(2b)^2 = 3$, and $(2c)^2 = \frac{5}{3}$, given by Lee (Ref. 3) for $SU(3)$.

[25] For $SU(3)$, this is explicitly demonstrated by Kuriyan *et al.* (Ref. 23) and noted by Hecht (Ref. 13).

B. Symmetry Relations

1. Exchange States 1, 2. The relations

$$f(\alpha, \beta, \gamma) = -f(\beta, \alpha, \gamma), \quad d(\alpha, \beta, \gamma) = d(\beta, \alpha, \gamma), \tag{32}$$

follow directly from Eqs. (22).

2. Exhange States 1, 3.

$$bf(\alpha, \beta, \gamma) = i\epsilon_i^{(\alpha)}\epsilon_j^{(\beta)}(-1)^{Q_\gamma}\epsilon_k^{(-\gamma)}f_{ijk} \quad \text{using Eq. (14),}$$

$$= -i\epsilon_i^{(-\gamma)}\epsilon_j^{(\beta)}\epsilon_k^{(\alpha)}(-1)^{Q_\gamma}f_{ijk} \quad \text{exchanging } i, k,$$

$$bf(-\gamma, \beta, -\alpha) = -i\epsilon_i^{(\alpha)}\epsilon_j^{(\beta)}\epsilon_k^{(-\gamma)}f_{ijk}(-1)^{Q_{-\alpha}}$$

$$= -(-1)^{Q_{-\alpha}+Q_{-\gamma}}f(\alpha, \beta, \gamma) \quad \text{using Eq. (14).}$$

Hence, under the exchange of states 1, 3,

$$f(-\gamma, \beta, -\alpha) = -(-1)^{Q_\beta}f(\alpha, \beta, \gamma), \quad d(-\gamma, \beta, -\alpha) = (-1)^{Q_\beta}\, d(\alpha, \beta, \gamma), \tag{33}$$

since $(-1)^{Q_{-\alpha}+Q_{-\gamma}} = (-1)^{Q_\beta}$.

3. Complex Conjugation.

$$bf(\alpha, \beta, \gamma) = bf^*(\alpha, \beta, \gamma), \quad \text{because } f \text{ is real,}$$

$$= -i\epsilon_i^{(\alpha)*}\epsilon_j^{(\beta)*}\epsilon_k^{(\gamma)}f_{ijk}$$

$$= (-1)(-1)^{Q_\alpha+Q_\beta-Q_\gamma}\epsilon_i^{(-\alpha)}\epsilon_j^{(-\beta)}(\epsilon_k^{(-\gamma)})^*f_{ijk} \quad \text{using Eq. (14).}$$

Hence,

$$f(-\alpha, -\beta, -\gamma) = -f(\alpha, \beta, \gamma), \quad d(-\alpha, -\beta, -\gamma) = d(\alpha, \beta, \gamma), \tag{34}$$

since $(-1)^{Q_\alpha+Q_\beta-Q_\gamma} = +1$.

IV. PRODUCTS OF GELL-MANN MATRICES

Notation: Let

$$(\alpha\beta\gamma)_r = \begin{cases} f(\alpha, \beta, \gamma) & r = 1 \\ d(\alpha, \beta, \gamma) & r = 2, \end{cases} \tag{35}$$

$$a_{(r)} = \begin{cases} a_f & r = 1 \\ a_d & r = 2, \end{cases} \tag{36}$$

and finally, let

$$C_k^{(r)} = \begin{cases} F_k & r = 1 \\ D_k & r = 2. \end{cases} \tag{37}$$

Note, from Eq. (37), that

$$(C_k^{(r)})^* = (C_k^{(r)})^T = (-1)^r C_k^{(r)}. \tag{38}$$

Using Eqs. (35) and (37), we may put Eq. (22) in the short-hand form

$$(C_i^{(r)})_{jk} = (a_{(r)})^{\frac{1}{2}}\epsilon_i^{(\nu_1)*}\epsilon_j^{(\nu_2)*}\epsilon_k^{(\nu_3)}(\nu_1\nu_2\nu_3)_r. \tag{39}$$

Consider the product

$$(C_m^{(r_1)}C_i^{(r_2)}C_m^{(r_3)})_{jk} = \alpha(F_i)_{jk} + \beta(D_i)_{jk} \tag{40a}$$

$$= \alpha(r, r_i)(C_i^{(r)})_{jk}, \tag{40b}$$

where $\alpha(1, r_i) \equiv \alpha$, $\alpha(2, r_i) \equiv \beta$. Multiply Eq. (40a) by $C_i^{(r)}$ and sum i,

$$\alpha(r, r_i) = (Na_{(r)})^{-1}\operatorname{Tr}(C_i^{(r)}C_m^{(r_1)}C_i^{(r_2)}C_m^{(r_3)}) \tag{41}$$

(r, not summed). Equation (41) expresses the coefficients $\alpha(r, r_i)$ as a trace of a product of four $C_r^{(k)}$. The coefficients $\alpha(r, r_i)$ are functions of r, r_1, r_2, r_3, and the order of the indices m, i, m, and not a function of the row or column labels of the matrices. The coefficients are therefore similar in construction to the 6-j symbols. To see this more clearly, we may express $\alpha(r, r_i)$ as a function of Wigner coefficients.

Using Eq. (39), we have

$$\alpha(r, r_i) = \frac{\prod_{i=1}^{3}(a_{(r_i)})^{\frac{1}{2}}}{N(a_{(r)})^{\frac{1}{2}}}\sum_{\nu_i} f_{\nu_i}\prod_{k=1}^{4}(\text{W.C.})_k, \tag{42a}$$

where

$$\prod_{k=1}^{4}(\text{W.C.})_k = (\nu_1\nu_2\nu_3)_r(\nu_4\nu_5\nu_6)_{r_1}(\nu_7\nu_8\nu_9)_{r_2}(\nu_{10}\nu_{11}\nu_{12})_{r_3}, \tag{42b}$$

and

$$f_{\nu_i} = \sum_{\alpha_k, i, m}(\epsilon_{\alpha_1}^{(\nu_1)*}\epsilon_{\alpha_2}^{(\nu_2)*}\epsilon_i^{(\nu_3)})(\epsilon_{\alpha_2}^{(\nu_4)*}\epsilon_{\alpha_3}^{(\nu_5)*}\epsilon_m^{(\nu_6)}) \times (\epsilon_{\alpha_3}^{(\nu_7)*}\epsilon_{\alpha_4}^{(\nu_8)*}\epsilon_i^{(\nu_9)})(\epsilon_{\alpha_4}^{(\nu_{10})*}\epsilon_{\alpha_1}^{(\nu_{11})*}\epsilon_m^{(\nu_{12})}). \tag{42c}$$

The summation indices α_i are the row, column indices of the matrices $C_i^{(r)}$. Employing the orthogonality relations, Eq. (15c), f_{ν_i} may be written

$$f_{\nu_i} = \delta_{\nu_1,-\nu_{11}}\delta_{\nu_2,-\nu_4}\delta_{\nu_5,-\nu_7}\delta_{\nu_8,-\nu_{10}}\delta_{\nu_3,-\nu_9}\delta_{\nu_6,-\nu_{12}}(-1)^{\phi}, \tag{43a}$$

where

$$\phi = -\nu_1 - \nu_2 - \nu_5 - \nu_8 + \nu_3 + \nu_6, \tag{43b}$$

and we have defined $(-1)^{\nu} \equiv (-1)^{Q_\nu}$. Equation (42a) then becomes

$$\alpha(r, r_i) = \frac{\prod_{i=1}^{3}(a_{(r_i)})^{\frac{1}{2}}}{N(a_r)^{\frac{1}{2}}}\sum_{\nu_i}(\nu_1\nu_2\nu_3)_r(-\nu_2\nu_5 - \nu_3)_{r_1} \times (-\nu_5\nu_8\nu_9)_{r_2}(-\nu_8 - \nu_1 - \nu_9)_{r_3}(-1)^{\phi}. \tag{44}$$

Using the triangle conditions on the Wigner coefficients, the symmetry relations and renaming the indices, we have

$$\alpha(r, r_i) = \frac{\prod_{i=1}^{3}(a_{(r_i)})^{\frac{1}{2}}}{N(a_r)^{\frac{1}{2}}}(-1)^{r_1+r_2}\sum_{\nu_i}(\nu_1\nu_2\nu_{12})_r(\nu_{12}\nu_3\nu)_{r_2} \times (\nu_1\nu_3\nu_{13})_{r_3}(\nu_{13}\nu_2\nu)_{r_1}, \tag{45a}$$

or

$$\alpha(r, r_i) = \prod_{i=1}^{3} (a_{(r_i)})^{\frac{1}{2}}(-1)^{r_1+r_2}(a_{(r)})^{-\frac{1}{2}}N[6\text{-}j], \quad (45b)$$

where $[6\text{-}j]$ is the 6-j coefficient which relates the coupling schemes

$$[((N_1) \times (N_2))_r \times (N_3)]_{r_2},$$
$$[((N_1) \times (N_3))_{r_3} \times (N_2)]_{r_1},$$

as can be seen directly from Eq. (45a). If the coefficients $\alpha(r, r_i)$ are known, then Eq. (45b) determines the 6-j coefficients for $SU(n)$. The coefficients are calculated in Appendix A, and given in Table I. For the case $SU(3)$, Eq. (45b) gives the crossing matrices determined by de Swart.[26] Crossing matrices for $SU(n)$[27] and semisimple Lie groups,[28] have been discussed by other authors.

Higher-order products of matrices of the regular representation may similarly be expressed in terms of a trace of a product of matrices, and then as a product of Wigner coefficients. Consider,

$$(C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_i^{(r_3)}C_{\mu_1}^{(r_4)}C_{\mu_2}^{(r_5)})_{jk} = \alpha(r, r_j)C_i^{(r)}. \quad (46)$$

Then,

$$\alpha(r, r_j) = (Na_{(r)})^{-1}\,\mathrm{Tr}\,(C_i^{(r)}C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_i^{(r_3)}C_{\mu_1}^{(r_4)}C_{\mu_2}^{(r_5)}) \quad (47)$$

with r, not summed. Using Eq. (41), this may be expressed in terms of Wigner coefficients,

$$\alpha(r, r_i) = \frac{\prod_{i=1}^{5} (a_{r_i})^{\frac{1}{2}} \cdot N^5}{(a_r)^{\frac{1}{2}}}\left[\frac{1}{N^6}\sum_{\nu i}(-1)^{\phi}\prod_{k=1}^{6}(\text{W.C.})_k\right], \quad (48a)$$

where

$$\phi = -\nu_1 - \nu_2 - \nu_5 - \nu_8 - \nu_{11} - \nu_{14} + \nu_3 + \nu_6 + \nu_9, \quad (48b)$$

and

$$\prod_{k=1}^{6}(\text{W.C.})_k = (\nu_1\nu_2\nu_3)_r(-\nu_2\nu_5\nu_6)_{r_1}(-\nu_5\nu_8\nu_9)_{r_2} \times (-\nu_8\nu_{11} - \nu_3)_{r_3}(-\nu_{11}\nu_{14} - \nu_6)_{r_4} \times (-\nu_{14} - \nu_1 - \nu_9)_{r_5}. \quad (48c)$$

The term in brackets in Eq. (48a) is, within a phase, the 9-j coefficient for $SU(n)$. Renaming the indices, and using the symmetry relations, we find that the term in brackets is the 9-j coefficient which relates the coupling schemes

$$[\ ] = (-1)^{r_3+r_4}9\text{-}j\begin{Bmatrix}[(1 \times 2)_r \times (3 \times 4)_{r_4}]_{r_3}\\ [(1 \times 4)_{r_5} \times (2 \times 3)_{r_1}]_{r_2}\end{Bmatrix}, \quad (49a)$$

$$[\ ] = (-1)^{r_1+r_3+r_5}9\text{-}j\begin{Bmatrix}[[(1 \times 2)_r \times 3]_{r_3} \times 4]_{r_2}\\ [[(1 \times 4)_{r_5} \times 3]_{r_4} \times 2]_{r_1}\end{Bmatrix}. \quad (49b)$$

The coefficients for the product of five matrices, Eq. (46), are given in Table II.

The product

$$C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_i^{(r_3)}C_{\mu_2}^{(r_4)}C_{\mu_1}^{(r_5)} = \alpha(r, r_i)C_i^{(r)} \quad (50)$$

with the last two indices μ_1, μ_2 reversed, compared to the product, Eq. (46), may easily be evaluated using the relations of Table I for the third-order product. They are not reproduced here. The product Eq. (46) is, within the same factors given in Eq. (48a), the 9-j coefficient which relates the coupling schemes

$$[\ ] = (-1)^{r_3+r_4}9\text{-}j\begin{Bmatrix}[(1 \times 2)_r \times (3 \times 4)_{r_4}]_{r_3}\\ [[(1 \times 4)_{r_5} \times 2]_{r_1} \times 3]_{r_2}\end{Bmatrix}, \quad (51a)$$

$$[\ ] = (-1)^{r_2+r_3}9\text{-}j\begin{Bmatrix}[[(1 \times 2)_r \times 3]_{r_3} \times 4]_{r_2}\\ [[(4 \times 2)_{r_2} \times 1]_{r_5} \times 3]_{r_4}\end{Bmatrix}. \quad (51b)$$

The other possible *fifth*-order matrix products, $C_{u_1}C_iC_{u_2}C_{u_1}C_{u_2}$, $C_{u_1}C_iC_{u_1}C_{u_2}C_{u_2}$, $C_{u_1}C_iC_{u_2}C_{u_2}C_{u_1}$, $C_iC_{u_1}C_{u_2}C_{u_2}C_{u_1}$ may also be evaluated from the third-order results.

Finally, the 12-j coefficients are given in Table III, and arise from the matrix products

$$C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_{\mu_3}^{(r_3)}C_i^{(r_4)}C_{\mu_1}^{(r_5)}C_{\mu_2}^{(r_6)}C_{\mu_3}^{(r_7)} = \alpha(r, r_i)C_i^{(r)},$$

$$\alpha(r, r_i) = (Na_{(r)})^{-1}\,\mathrm{Tr}\,(C_i^{(r)}C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_{\mu_3}^{(r_3)}C_i^{(r_4)} \times C_{\mu_1}^{(r_5)}C_{\mu_2}^{(r_6)}C_{\mu_3}^{(r_7)}). \quad (52)$$

The coefficients $\alpha(r, r_i)$, in terms of Wigner symbols, may be written

$$\alpha(r, r_i) = \frac{\prod_{i=1}^{7} (a_{(r_i)})^{\frac{1}{2}}}{N(a_{(r)})^{\frac{1}{2}}}\left[\sum_{\nu_i}(-1)^{\phi}\prod_{k=1}^{8}(\text{W.C.})_k\right], \quad (53)$$

where

$$(-1)^{\phi} = -\nu_1 - \nu_2 - \nu_5 - \nu_8 - \nu_{11} - \nu_{14} - \nu_{17} - \nu_{20} + \nu_3 + \nu_6 + \nu_9 + \nu_{12}$$

[26] J. J. de Swart, Nuovo Cimento **31**, 420 (1961). The above (6-j) divided by N^2 is de Swart's crossing matrix.

[27] C. L. Cook, G. Murtaza, and M. A. Rashid, Nuovo Cimento **41**, 122 (1966); H. S. Mani, G. Mohan, L. K. Pande, and V. Singh, Ann. Phys. **36**, 285 (1966). See also, D. B. Fairlie, The Structure of the Crossing Matrix for Arbitrary Internal Symmetry Groups—II Matrices in $SU(n)$, University of Durham (1966), and J. F. L. Hopkinson, Imperial College, 1966.

[28] D. B. Fairlie, J. Math. Phys. **7**, 811 (1966).

and

$$\prod_{k=1}^{8} (\text{W.C.})_k = (\nu_1\nu_2\nu_3)_r(-\nu_2\nu_5\nu_6)_{r_1}(-\nu_5\nu_8\nu_9)_{r_2}$$
$$\times (-\nu_8\nu_{11}\nu_{12})_{r_3}(-\nu_{11}\nu_{14} - \nu_3)_{r_4}(-\nu_{14}\nu_{17} - \nu_6)_{r_5}$$
$$\times (-\nu_{17}\nu_{20} - \nu_9)_{r_6}(-\nu_{20} - \nu_1 - \nu_{12})_{r_7}.$$

The bracket in Eq. (53), times the factor $(N)^{-8}$ is the 12-j coefficient which relates the various ways the product of five regular representations may be coupled to yield the regular representation, as may be seen by appropriately relabelling the summation indices ν_i, and using the symmetry relations of the Wigner coefficients.

The product

$$(C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_{\mu_3}^{(r_3)}C_i^{(r_4)}C_{\mu_3}^{(r_5)}C_{\mu_2}^{(r_6)}C_{\mu_1}^{(r_7)}) \tag{54}$$

may easily be evaluated from the third-order results. All other products of seven matrices may be reduced to Eqs. (52) or Eq. (54) plus a term which is of *fifth* order in the matrix products. For example, consider the product

$$M_i = C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_{\mu_3}^{(r_3)}C_i^{(r_4)}C_{\mu_1}^{(r_5)}C_{\mu_3}^{(r_6)}C_{\mu_2}^{(r_7)}$$
$$= C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_{\mu_3}^{(r_3)}C_i^{(r_4)}C_{\mu_1}^{(r_5)}[C_{\mu_3}^{(r_6)}, C_{\mu_2}^{(r_7)}] + \alpha(r, r_i)C_i^{(r)},$$

using Eq. (52), the $\alpha(r, r_i)$ known from Table III.

$$M_i = \tfrac{1}{2}C_{\mu_1}^{(r_1)}[C_{\mu_2}^{(r_2)}, C_{\mu_3}^{(r_2)}]C_i^{(r_4)}C_{\mu_1}^{(r_5)}[C_{\mu_3}^{(r_6)}, C_{\mu_2}^{(r_7)}]$$
$$+ \alpha(r, r_i)C_i^{(r)}.$$

The first term may be reduced to the product of five matrices using the commutation results, Eqs. (A2), (A4), (A6), and the orthogonality relations, Eq. (8). These products provide other 12-j coefficients; the coupling schemes related by such coefficients may be found in the same manner as above.

Three features of the tables should be noted. The product of 3, 5, or 7 matrices of the regular representation, summed (as in the Tables) so as to leave one index free, yields an F_i or D_i, but not both. Thus, of the $2^4 = 16$ possible 6-j coefficients for the product Eq. (40a), we have only half that number, eight, which are nonzero. Similarly, there are $2^5 = 32$ possible 9-j coefficients for a given ordering of the indices in the product of five matrices, and $2^7 = 128$ 12-j coefficients for the product of seven matrices, as given in the Tables. Further, an odd number of F's in the product always yields an αF_i, and an even number of F's (an odd number of D's) always yields a βD_i. Finally, note that the coefficients α, β are always real. These results hold not only for the products in the tables, but for any permutation of the indices of the products in the tables.

Moreover, these results actually hold for a general product of Gell-Mann matrices F_i, D_i, and are a consequence of the charge conjugation properties of the regular representation matrices, as we now show. Consider the product

$$M_i = C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)} \cdots C_{\mu_p}^{(r_p)}C_i^{(r_{p+1})}C_{\mu_1'}^{(r_{p+2})} \cdots C_{\mu_p'}^{(r_{2p+1})}$$
$$= \alpha F_i + \beta D_i = \alpha(r, r_i)C_i^{(r)}, \tag{55}$$

where $\mu_i' \cdots \mu_p'$ is a permutation of the indices $\mu_1 \cdots \mu_p$; or

$$\alpha(r, r_i) = (Na_{(r)})^{-1} \operatorname{Tr}(C_i^{(r)}M_i). \tag{56}$$

Take the complex conjugate of Eq. (54),

$$\alpha^*(r, r_i) = (-1)^\theta \alpha(r, r_i), \tag{57}$$

where $\theta = r + r_1 + \cdots + r_{2p+1}$, using Eq. (38). Employing Eq. (39), we may express Eq. (56) in terms of Wigner coefficients

$$\alpha(r, r_i) = \frac{\prod_{i=1}^{2p+1}(a_{r_i})^{\frac{1}{2}}}{N(a_r)^{\frac{1}{2}}} \sum_{\nu_i} f_{\nu_i} \prod_{k=1}^{2p+2}(\text{W.C.})_k, \tag{58a}$$

where

$$f_{\nu_i} = \sum_{\alpha_j, \mu_k, i} (\epsilon_{\alpha_1}^{(\nu_1)*}\epsilon_{\alpha_2}^{(\nu_2)*}\epsilon_i^{(\nu_3)})$$
$$\times (\epsilon_{\alpha_2}^{(\nu_4)*}\epsilon_{\alpha_3}^{\nu_5)*}\epsilon_{\mu_1}^{(\nu_6)}) \cdots (\epsilon_{\alpha_{2p+2}}^{(\nu_{6p+4})*}\epsilon_{\alpha_1}^{(\nu_{6p+5})*}\epsilon_{\mu_p'}^{(\nu_{6(p+1)})}), \tag{58b}$$

and where the α_j are the row and column labels on the matrices $C_k^{(r_l)}$ appearing in Eq. (56) [and Eq. (55)], and $i\mu_1 \cdots \mu_p i\mu_1' \cdots \mu_p'$ are the subindices appearing in Eq. (55). We note that because Eq. (56) is the trace, to each $\epsilon_{\alpha_i}^{(\nu_k)*}$, there exists an $\epsilon_{\alpha_i}^{(\nu_{k'})*}$, summed on α_i. However,

$$\epsilon_{\alpha_i}^{(\nu_k)*}\epsilon_{\alpha_i}^{(\nu_{k'})*} = \delta_{\nu_k, -\nu_{k'}}(-1)^{(-\nu_k)},$$

using Eq. (18c). Further, to each $\epsilon_{\mu_s}^{(\nu_k)}$, there exists an $\epsilon_{\mu_{r'}}^{(\nu_{k'})}$, where $\mu_r' = \mu_s$ because $\mu_1' \cdots \mu_p'$ is only a permutation of the indices $\mu_1 \cdots \mu_p$. Summed on μ_s, we have

$$\epsilon_{\mu_s}^{(\nu_k)}\epsilon_{\mu_s}^{(\nu_{k'})} = \delta_{\nu_k, -\nu_{k'}}(-1)^{\nu_k}.$$

Finally, to $\epsilon_i^{(\nu_3)}$, there exists an $\epsilon_i^{\nu_{3p+6}}$, summed on i. Hence, f_{ν_i} is a product of Kronecker deltas times a real phase $(-1)^\phi$. Since the Wigner coefficients are real [see Eqs. (17) and (18)], the coefficients $\alpha(r, r_i)$ are real; $\alpha^*(r, r_i) = \alpha(r, r_i)$. This says the phase $(-1)^\phi = +1$, and θ must be even. If $r_1 + \cdots + r_{2p+1}$ is even, r is even; if $r_1 + \cdots + r_{2p+1}$ is odd, r is odd. We have the result then that an even number of F's in the product, Eq. (55), yields a βD_i and an odd number of F's in the product yields an αF_i, where α, β are real. This proof does not depend on the order of the subindices in Eq. (55), only that to each $C_{\mu_k}^{(r_s)}$, there exists a partner $C_{\mu_m}^{(r_t)}$ (or $C_{\mu_m}^{(r_t)}) = C_{\mu_m}^{(r_t)}$, summed on μ_k. Further C_i may be placed anywhere in the product and the result still holds.

From the charge conjugation symmetry property of the Wigner coefficients, Eq. (38), the proof may be carried through using the Wigner coefficients instead of the matrices $C_k^{(vs)}$. The above result shows that in any $(3p)$-j coefficient there must be an even number of antisymmetric Wigner coefficients which occur in the product of $2p$ Wigner coefficients in the $(3p)$-j symbol.

APPENDIX A: CALCULATION OF MATRIX PRODUCTS

From Eq. (2a), defining the Lie algebra, and the definition $(F_i)_{jk} = -if_{ijk}$, we obtain from the first Jacobi identity

$$[[\lambda_i, \lambda_j], \lambda_k] + [[\lambda_j, \lambda_k], \lambda_i] + [[\lambda_k, \lambda_i], \lambda_j] = 0, \tag{A1}$$

the result

$$[F_i, F_j] = if_{ijk}F_k. \tag{A2}$$

From the second Jacobi identity

$$[\{\lambda_i, \lambda_j\}, \lambda_k] + [\{\lambda_j, \lambda_k\}, \lambda_i] + [\{\lambda_k, \lambda_i\}; \lambda_j] = 0 \tag{A3}$$

and Eqs. (2) and $(D_i)_{jk} = d_{ijk}$, we have the commutation relations

$$[F_i, D_j] = [D_i, F_j] = if_{ijk}D_k, \tag{A4}$$

and the relations

$$D_iF_j + D_jF_i = F_iD_j + F_jD_i = d_{ijk}F_k. \tag{A5}$$

Since the coefficients f_{ijk} are antisymmetric in all indices [see Eq. (3a)], the matrices F_i are traceless. Further, since the trace of a commutator vanishes, we see from Eq. (A4) that the matrices D_i must be traceless also.

From the identity

$$[\lambda_i, [\lambda_j, \lambda_k]] = \{\lambda_k, \{\lambda_i, \lambda_j\}\} - \{\lambda_j, \{\lambda_i, \lambda_k\}\},$$

we have the commutation relation for the matrices D_i,

$$[D_i, D_j]_{mn} = if_{ijk}(F_k)_{mn} + a(\delta_{mj}\delta_{ni} - \delta_{im}\delta_{jn}). \tag{A6}$$

Finally, from the identity

$$[\lambda_j, [\lambda_k, \lambda_i]] - [\lambda_i, [\lambda_j, \lambda_k]] = \{\lambda_i, \{\lambda_j, \lambda_k\}\} + \{\lambda_j, \{\lambda_i, \lambda_k\}\} - 2\{\lambda_k, \{\lambda_i, \lambda_j\}\},$$

we obtain the anticommutation relations

$$\{D_i, D_j\}_{mn} + \{F_i, F_j\}_{mn} = 2a\delta_{ij}1 + 2d_{ijk}(D_k)_{mn} - a(\delta_{mj}\delta_{ni} + \delta_{mi}\delta_{nj}). \tag{A7}$$

If we add Eqs. (A2), (A6), and (A7), we get for the sum of the products

$$(F_iF_j + D_iD_j)_{mn} = a\delta_{ij}1 + d_{ijk}(D_k)_{mn} + if_{ijk}(F_k)_{mn} - a\delta_{im}\delta_{jn}. \tag{A8}$$

Equations (A2), (A4)–(A7) represent six linearly independent relations which can readily be obtained from the associativity condition

$$(\lambda_i\lambda_j)\lambda_k = \lambda_i(\lambda_j\lambda_k).$$

The six equations (A2), (A4)–(A7), are the only relations uniquely defined since the associativity condition provides only six relations. However, the products F_iF_j, D_iD_j, F_iD_j, D_iF_j, require eight relations, hence, they are not uniquely defined in the general case. The above relations are sufficient to evaluate the products of Gell-Mann matrices.

To begin, from Eq. (A6), we have

$$\mathrm{Tr}\,(F_iD_jD_k) = \tfrac{1}{2}f_{ijm}\,\mathrm{Tr}\,(F_mF_k) + iaf_{ijk}.$$

However, from Eq. (A4), we have

$$\mathrm{Tr}\,(F_iD_jD_k) = \tfrac{1}{2}f_{ijm}\,\mathrm{Tr}\,(D_mD_k). \tag{A9}$$

Subtracting these two equations, we obtain

$$f_{ijm}[\mathrm{Tr}\,(F_kF_m) - \mathrm{Tr}\,(D_kD_m)] = 2af_{ijk}.$$

From Eq. (A8), it follows that

$$f_{ijm}[\mathrm{Tr}\,(F_kF_m) + \mathrm{Tr}\,(D_kD_m)] = a(N-1)f_{ijk}.$$

Combining these two equations, we find

$$\begin{aligned}\mathrm{Tr}\,(F_kF_m)f_{ijm} &= a_ff_{ijk},\\ \mathrm{Tr}\,(D_kD_m)f_{ijm} &= a_df_{ijk},\end{aligned} \tag{A10}$$

where $a_f = \frac{1}{2}a(N+1)$ and $a_d = \frac{1}{2}a(N-3)$. This implies, since it holds for all i, j, k, and since the matrices F_i of the regular representation are independent, that

$$\begin{aligned}\mathrm{Tr}\,(F_iF_j) &= a_f\delta_{ij},\\ \mathrm{Tr}\,(D_iD_j) &= a_d\delta_{ij}.\end{aligned} \tag{A11}$$

To evaluate triple products of Gell-Mann matrices, let C_k be defined as in Eq. (37). Then we may write

$$(C_m^{(r_1)}C_i^{(r_2)}C_m^{(r_3)})_{jk} = \mathrm{Tr}\,(C_i^{(r_2)T}C_j^{(r_1)}C_k^{(r_3)}) = (-1)^{r_2}\,\mathrm{Tr}\,(C_i^{(r_2)}C_j^{(r_1)}C_k^{(r_3)}). \tag{A12}$$

It is necessary, then, to evaluate traces of products of matrices of the regular representation. From Eqs. (A11) and (A2),

$$\mathrm{Tr}\,(F_iF_jF_k) = \tfrac{1}{2}a_ff_{ijk}. \tag{A13}$$

From Eqs. (A9) and (A2),

$$\mathrm{Tr}\,(F_iD_jD_k) = \tfrac{1}{2}a_df_{ijk}. \tag{A14}$$

TABLE I. States, weights of the $SU(n)$ regular representation.

State (in)	h_1	h_2	h_3	h_4	h_5	h_6	h_{n-2}	h_{n-1}
$(1n)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$1/n-1$	1
$(2n)$	$-\frac{1}{2}$	$\frac{1}{3}$						
$(3n)$	0	$-\frac{2}{3}$	$\frac{1}{4}$					
$(4n)$	0	0	$-\frac{3}{4}$	$\frac{1}{5}$				
$(5n)$	0	0	0	$-\frac{4}{5}$	$\frac{1}{6}$			
$(6n)$	0	0	0	0	$-\frac{5}{6}$	$\frac{1}{7}$	$1/n-1$	1
$(n-2n)$	0	0	0	0	0		$1/n-1$	1
$(n-1n)$	0	0	0	0	0	0	$\frac{-(n-2)}{(n-1)}$	1

Using Eq. (A4), it may be shown that

$$\operatorname{Tr}(F_iD_jD_k) = \operatorname{Tr}(D_iF_jD_k) = \operatorname{Tr}(D_iD_jF_k), \tag{A15}$$

$$\operatorname{Tr}(F_iF_jD_k) = \operatorname{Tr}(F_iD_jF_k) = \operatorname{Tr}(D_iF_jF_k). \tag{A16}$$

Then, Eqs. (A13), (A14), and (A10) taken together, yield

$$\operatorname{Tr}(D_iF_jF_k) = \tfrac{1}{2}a_f\, d_{ijk}. \tag{A17}$$

Finally, from Eq. (A8), we have

$$\operatorname{Tr}(F_iF_jD_k) + \operatorname{Tr}(D_iD_jD_k) = \tfrac{1}{2}a(N-5)\, d_{ijk}. \tag{A18}$$

Using Eqs. (A16) and (A17), we obtain

$$\operatorname{Tr}(D_iD_jD_k) = \tfrac{1}{2}b_d\, d_{ijk}, \tag{A19}$$

where $b_d = \frac{1}{2}a(N-11)$. From the above trace equations and (A12), we get the results of Table I.

The products of five Gell-Mann matrices follow in a similar manner. The analog of Eq. (A12) is

$$\begin{aligned}(C_{\mu_1}^{(r_1)}C_{\mu_2}^{(r_2)}C_i^{(r_3)}C_{\mu_1}^{(r_4)}C_{\mu_2}^{(r_5)})_{jk} &= \operatorname{Tr}(C_i^{(r_3)T}C_m^{(r_2)T}C_j^{(r_1)}C_n^{(r_4)})(C_k^{(r_5)})_{mn} \\ &= \operatorname{Tr}(C_j^{(r_1)T}C_m^{(r_2)T}C_k^{(r_5)}C_n^{(r_4)})(C_i^{(r_3)})_{mn} \\ &= \operatorname{Tr}(C_k^{(r_5)T}C_m^{(r_2)T}C_i^{(r_3)}C_k^{(r_4)})(C_j^{(r_1)})_{mn}.\end{aligned} \tag{A20}$$

Two quadratic relations are useful in obtaining the results of Table II:

$$\begin{aligned}&\operatorname{Tr}(F_iF_jF_kD_m) - \operatorname{Tr}(F_jF_iF_kD_m) \\ &\quad = \operatorname{Tr}(F_iF_jF_kD_m) + \operatorname{Tr}(F_iF_jD_mF_k) = \tfrac{1}{2}a_f f_{ijn}\, d_{nkm}\end{aligned}$$

using Eqs. (A4) and (A17). But, from Eqs. (A4) and (A17), we have

$$\operatorname{Tr}(F_iF_jF_kD_m) - \operatorname{Tr}(F_iF_jD_mF_k) = \tfrac{1}{2}a_f f_{kmn}\, d_{ijn}.$$

Adding these two equations gives the relation

$$\operatorname{Tr}(F_iF_jF_kD_m) = \tfrac{1}{4}ia_f(f_{ijn}\, d_{nkm} + f_{kmn}\, d_{ijn}). \tag{A21}$$

In a similar manner, from Eqs. (A4), (A6), (A17), and (A19), we obtain the second relation

$$\begin{aligned}\operatorname{Tr}(F_iD_jD_kD_m) &= \frac{i}{4}(b_d f_{ijn}\, d_{nkm} + a_f f_{kmn}\, d_{ijn}) \\ &\quad + \tfrac{1}{2}ia(f_{imn}\, d_{jnk} - f_{ikn}\, d_{jnm}).\end{aligned} \tag{A22}$$

Using the above two relations, we may evaluate the trace terms as required by (A20). We note immediately that

$$\operatorname{Tr}(F_iF_mF_jF_n)f_{kmn} = 0, \tag{A23}$$

$$\operatorname{Tr}(D_iF_mD_jF_n)f_{kmn} = 0, \tag{A24}$$

$$\operatorname{Tr}(F_iD_mF_jD_n)f_{kmn} = 0, \tag{A25}$$

$$\operatorname{Tr}(D_iD_mD_jD_n)f_{kmn} = 0, \tag{A26}$$

because, in all these cases, the trace is symmetric with respect to the interchange of the indices m, n, whereas f_{kmn} is antisymmetric. Further,

$$\operatorname{Tr}(F_iF_mF_jD_n)f_{kmn} = 0, \tag{A27}$$

using Eqs. (A21) and (A16).

$$\operatorname{Tr}(F_iF_mF_jD_n)\, d_{kmn} = 0, \tag{A28}$$

using Eqs. (A21) and (A15).

$$\operatorname{Tr}(F_iF_mD_jF_n)f_{kmn} = 0, \tag{A29}$$

using Eqs. (A4), (A17), and (A21).

$$\operatorname{Tr}(F_iF_mD_jF_n)\, d_{kmn} = 0, \tag{A30}$$

TABLE II. Third-order products.

$$\begin{aligned}(F_mF_iF_m) &= \tfrac{1}{2}a_fF_i \\ F_mF_iD_m = D_mF_iF_m &= -\tfrac{1}{2}a_fD_i \\ F_mD_iF_m &= \tfrac{1}{2}a_fD_i \\ (F_mD_iD_m) = D_mD_iF_m &= -\tfrac{1}{2}a_dF_i \\ D_mF_iD_m &= \tfrac{1}{2}a_dF_i \\ D_mD_iD_m &= \tfrac{1}{2}b_dD_i\end{aligned}$$

$$a_f = \tfrac{1}{2}a(N+1),\quad a_d = \tfrac{1}{2}a(N-3),\quad b_d = \tfrac{1}{2}a(N-11)$$

TABLE III. Fifth-order products.[a]

All products of 5F's 4F's and 1D, 3F's and 2D's are zero.

$$D_mD_nD_iF_mF_n = D_mD_nF_iF_mD_n = 0$$
$$D_mD_nF_iD_mF_n = D_mF_nD_iF_mD_n = aa_fD_i$$
$$D_mD_nD_iD_mF_n = D_mD_nF_iD_mD_n = 0$$
$$D_mD_nD_iF_mD_n = aa_dF_i$$
$$D_mD_nD_iD_mD_n = -ac_dD_i \quad \text{where} \quad c_d = a(N-9),$$
$$a_f = \tfrac{1}{2}a(N+1) \quad \text{and} \quad a_d = \tfrac{1}{2}a(N-3).$$

[a] Transpose of the products in the table give the remaining possible fifth-order products.

TABLE IV. Seventh-order products.[a]

$$F_{\mu_1}F_{\mu_2}F_{\mu_3}F_iF_{\mu_1}F_{\mu_2}F_{\mu_3} = a(\tfrac{1}{2}a)^2$$

All products of 6F's, 1D and 5F's, 2D's are zero.

$$\begin{aligned}F_{\mu_1}F_{\mu_2}F_{\mu_3}F_iD_{\mu_1}D_{\mu_2}D_{\mu_3} &= -F_{\mu_1}F_{\mu_2}F_{\mu_3}D_iD_{\mu_1}D_{\mu_2}F_{\mu_3} = -F_{\mu_1}F_{\mu_2}D_{\mu_3}D_iD_{\mu_1}F_{\mu_2}F_{\mu_3}\\
&= F_{\mu_1}F_{\mu_2}F_{\mu_3}D_iF_{\mu_1}D_{\mu_2}D_{\mu_3} = -F_{\mu_1}F_{\mu_2}D_{\mu_3}F_iD_{\mu_1}D_{\mu_2}F_{\mu_3} = -F_{\mu_1}D_{\mu_2}F_{\mu_3}D_iD_{\mu_1}F_{\mu_2}F_{\mu_3}\\
&= +D_{\mu_1}F_{\mu_2}D_{\mu_3}D_iF_{\mu_1}F_{\mu_2}F_{\mu_3} = -F_{\mu_1}F_{\mu_2}D_{\mu_3}F_iF_{\mu_1}D_{\mu_2}D_{\mu_3} = -F_{\mu_1}D_{\mu_2}F_{\mu_3}F_iD_{\mu_1}D_{\mu_2}F_{\mu_3}\\
&= D_{\mu_1}F_{\mu_2}F_{\mu_3}D_iD_{\mu_1}F_{\mu_2}F_{\mu_3} = F_{\mu_1}D_{\mu_2}F_{\mu_3}F_iF_{\mu_1}D_{\mu_2}D_{\mu_3} = D_{\mu_1}F_{\mu_2}F_{\mu_3}F_iD_{\mu_1}D_{\mu_2}F_{\mu_3}\\
&= -D_{\mu_1}F_{\mu_2}F_{\mu_3}F_iF_{\mu_1}D_{\mu_2}D_{\mu_3} = -F_{\mu_1}F_{\mu_2}D_{\mu_3}F_iD_{\mu_1}F_{\mu_2}D_{\mu_3} = F_{\mu_1}D_{\mu_2}F_{\mu_3}D_iF_{\mu_1}D_{\mu_2}F_{\mu_3}\\
&= -F_{\mu_1}D_{\mu_2}F_{\mu_3}F_iD_{\mu_1}F_{\mu_2}D_{\mu_3} = -D_{\mu_1}F_{\mu_2}F_{\mu_3}D_iF_{\mu_1}D_{\mu_2}F_{\mu_3} = D_{\mu_1}F_{\mu_2}F_{\mu_3}F_iD_{\mu_1}F_{\mu_2}D_{\mu_3}\\
&= -D_{\mu_1}F_{\mu_2}F_{\mu_3}D_iF_{\mu_1}F_{\mu_2}D_{\mu_3} = a(\tfrac{1}{2}a_f)^2D_i\end{aligned}$$

$$\begin{aligned}-D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iF_{\mu_1}F_{\mu_2}F_{\mu_3} &= D_{\mu_1}D_{\mu_2}D_{\mu_3}F_iF_{\mu_1}F_{\mu_2}D_{\mu_3} = D_{\mu_1}D_{\mu_2}F_{\mu_3}F_iF_{\mu_1}D_{\mu_2}D_{\mu_3}\\
&= D_{\mu_1}D_{\mu_2}D_{\mu_3}F_iD_{\mu_1}F_{\mu_2}F_{\mu_3} = -D_{\mu_1}D_{\mu_2}F_{\mu_3}D_iF_{\mu_1}F_{\mu_2}D_{\mu_3} = -D_{\mu_1}F_{\mu_2}D_{\mu_3}F_iF_{\mu_1}D_{\mu_2}D_{\mu_3}\\
&= -F_{\mu_1}D_{\mu_2}F_{\mu_3}F_iD_{\mu_1}D_{\mu_2}D_{\mu_3} = D_{\mu_1}D_{\mu_2}F_{\mu_3}D_iD_{\mu_1}F_{\mu_2}F_{\mu_3} = -D_{\mu_1}F_{\mu_2}D_{\mu_3}D_iF_{\mu_1}F_{\mu_2}D_{\mu_3}\\
&= F_{\mu_1}D_{\mu_2}D_{\mu_3}F_iF_{\mu_1}D_{\mu_2}D_{\mu_3} = D_{\mu_1}F_{\mu_2}D_{\mu_3}D_iD_{\mu_1}F_{\mu_2}F_{\mu_3} = -F_{\mu_1}D_{\mu_2}D_{\mu_3}D_iF_{\mu_1}F_{\mu_2}D_{\mu_3}\\
&= -F_{\mu_1}D_{\mu_2}D_{\mu_3}D_iD_{\mu_1}F_{\mu_2}F_{\mu_3} = -D_{\mu_1}D_{\mu_2}F_{\mu_3}D_iF_{\mu_1}D_{\mu_2}F_{\mu_3} = D_{\mu_1}F_{\mu_2}D_{\mu_3}F_iD_{\mu_1}F_{\mu_2}D_{\mu_3}\\
&= -D_{\mu_1}F_{\mu_2}D_{\mu_3}D_iF_{\mu_1}D_{\mu_2}F_{\mu_3} = -F_{\mu_1}D_{\mu_2}D_{\mu_1}F_iD_{\mu_1}F_{\mu_2}D_{\mu_3} = F_{\mu_1}D_{\mu_2}D_{\mu_3}D_iF_{\mu_1}D_{\mu_2}F_{\mu_3}\\
&= F_{\mu_1}D_{\mu_2}D_{\mu_3}F_iD_{\mu_1}D_{\mu_2}F_{\mu_3} = \frac{a}{4}\,(a_fa_d)\,F_i\end{aligned}$$

$$D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iD_{\mu_1}F_{\mu_2}F_{\mu_3} = D_{\mu_1}D_{\mu_2}D_{\mu_3}F_iF_{\mu_1}D_{\mu_2}D_{\mu_3} = -\frac{a}{4}\,(a_fe_d)\,D_i$$
$$\begin{aligned}D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iF_{\mu_1}F_{\mu_2}D_{\mu_3} &= D_{\mu_1}D_{\mu_2}D_{\mu_3}F_iD_{\mu_1}D_{\mu_2}F_{\mu_3} = D_{\mu_1}D_{\mu_2}F_{\mu_3}D_iD_{\mu_1}F_{\mu_2}D_{\mu_3}\\
&= D_{\mu_1}D_{\mu_2}F_{\mu_3}D_iD_{\mu_1}D_{\mu_2}F_{\mu_3} = D_{\mu_1}F_{\mu_2}D_{\mu_3}D_iD_{\mu_1}F_{\mu_2}D_{\mu_3} = \frac{+a}{4}(a_fc_d)D_i\end{aligned}$$
$$\begin{aligned}-D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iF_{\mu_1}D_{\mu_2}F_{\mu_3} &= -D_{\mu_1}D_{\mu_2}F_{\mu_3}D_iF_{\mu_1}F_{\mu_2}D_{\mu_3} = D_{\mu_1}F_{\mu_2}D_{\mu_3}D_iD_{\mu_1}D_{\mu_2}F_{\mu_3}\\
&= F_{\mu_1}D_{\mu_2}D_{\mu_3}D_iD_{\mu_1}D_{\mu_2}F_{\mu_3} = \tfrac{1}{2}(s)^2a_fD_i\end{aligned}$$
$$D_{\mu_1}D_{\mu_2}D_{\mu_3}F_iD_{\mu_1}F_{\mu_2}D_{\mu_3} = \tfrac{1}{2}aa_ff_dD_i$$

$$D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iD_{\mu_1}D_{\mu_2}F_{\mu_3} = \frac{-a}{4}\,(a_dc_d)F_i$$
$$D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iD_{\mu_1}F_{\mu_2}D_{\mu_3} = -\tfrac{1}{2}(a)^2a_dF_i$$
$$D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iF_{\mu_1}D_{\mu_2}D_{\mu_3} = D_{\mu_1}D_{\mu_2}D_{\mu_3}F_iD_{\mu_1}D_{\mu_2}D_{\mu_3} = \frac{a}{4}\,(a_dc_d)F_i$$
$$D_{\mu_1}D_{\mu_2}D_{\mu_3}D_iD_{\mu_1}D_{\mu_2}D_{\mu_3} = D_i\tfrac{1}{4}a((a_f)^2 - g_db_d)$$

where

$$e_d = a(N-5),\quad f_d = a(N-6),\quad c_d = a(N-9),\quad g_d = a(N-21)$$
$$a_f = \tfrac{1}{2}a(N+1),\quad a_d = \tfrac{1}{2}a(N-3),\quad b_d = \tfrac{1}{2}a(N-11)$$

[a] Transpose of the products in the table gives the remaining possible seventh-order products.

since the trace is antisymmetric under the interchange of m, n. Using Eq. (A20) and employing the symmetry of the matrix products, we see that all cases of the products containing one or two D's vanish.

In a similar manner, but now using Eq. (A22),

$$\mathrm{Tr}\,(F_iF_mD_jD_n)f_{kmn} = iaa_fd_{ijk}, \tag{A31}$$

also using Eq. (A17).

$$\mathrm{Tr}\,(D_iF_mD_jD_n)f_{kmn} = 0, \tag{A32}$$

using Eqs. (A4), (A19), and (A22).

$$\mathrm{Tr}\,(D_iF_mD_jD_n)d_{kmn} = iaa_df_{ijk}, \tag{A33}$$

using Eqs. (A4), (A14), (A19), and (A22). Combining Eqs. (A4) and (A5), we have the relation

$$F_kD_m + D_kF_m = d_{kmn}F_n + if_{kmn}D_n.$$

Using this relation and Eqs. (A17), (A19), (A24), and Table I, we have

$$\mathrm{Tr}\,(D_iF_mD_jF_n)d_{kmn} = -aa_fd_{ijk}. \tag{A34}$$

For the final relation, note that

$$\mathrm{Tr}\,(D_iD_mF_jF_n)d_{kmn} = i\,\mathrm{Tr}\,(F_nD_iD_mD_k)f_{jmn},$$

using (A20). Then, using Eqs. (A11), (A17), and (A22),

$$\mathrm{Tr}\,(D_iD_mF_jF_n)d_{kmn} = 0.$$

Finally, using this equation, and Eqs. (A8), (A11), (A17), and (A19), we obtain

$$\mathrm{Tr}\,(D_iD_mD_jD_n)d_{kmn} = -ac_dd_{ijk}, \tag{A35}$$

where $c_d = a(N - 9)$. These trace relations, together with Eq. (A20), give the results of Table II. Table III follows in a similar manner; the calculations are straightforward, but tedious and are not reproduced here.

APPENDIX B: PHASE CONVENTION OF WIGNER COEFFICIENTS

The phase convention, Eq. (26), and the generalized charge Q_α, Eq. (13d), lead to a positive phase for the matrix elements of the raising operators $L^{(1j)}$,

$$\langle\gamma|\,L^{(1j)}\,|\beta\rangle \geq 0, \quad j = 2, \cdots, n. \tag{B1}$$

It may be shown first, from Eq. (13d) and Table I, that

$$\begin{aligned} (-1)^{Q_{(1j)}} &= +1 \quad j = 2, \cdots, n, \\ (-1)^{Q_{(j1)}} &= -1 \quad \text{otherwise}, \\ (-1)^{Q_{(ij)}} &= 0. \end{aligned} \tag{B2}$$

Consider $\alpha = (1j)$, β, $\gamma = \pm 1, \cdots, \pm m/2$. Then, from Eq. (26b), $(-1)^\omega = +1$. From the triangle relations of Eqs. (B3) and (B4) we have

$$\alpha = (1j), \beta = (jl), \gamma = (1l), \quad j > 1, \tag{B3}$$

$$\alpha = (1j), \beta = (k1), \gamma = (kj), \quad j > 1. \tag{B4}$$

From Eq. (B3), since $j > 1$, and $l > 1$, $\theta = Q_\alpha + Q_\beta = 2$, hence, a positive phase. From Eq. (B4), $j > 1, k > 1$, and θ is 0, $(-1)^\theta = 1$. If $\alpha = (1j)$, and $\beta = -\alpha = (j1)$, from Eq. (17e), $2bf(\alpha, \beta, \gamma)$ is negative. However, $(-1)^\omega = -1$, by Eq. (26b), so the matrix element, Eq. (27), is positive. Finally, let $\alpha = (1j)$, $\beta = m + 1, \cdots, N$. Then, from Eq. (17c), $2bf(\alpha, \beta, \gamma)$ is negative. Here again, $(-1)^\omega = -1$, so Eq. (27) is positive.

Equation (27) and the conjunction operation, $L^{(\alpha)} = (-1)^{Q_\alpha}L^{(-\alpha)}$, determines the relative phases.[29] The absolute phase of the Wigner coefficient may be fixed by requiring the coefficient with states $\alpha_{\max} = \gamma_{\max} = (12)$ be positive. This coefficient is nonzero.

The relative phases of the coefficients

$$\begin{pmatrix} N & N & N \\ \alpha & \beta & \gamma \end{pmatrix}_2$$

are determined by the phase convention, Eq. (26). The absolute phase is fixed, as above. Note that for $SU(3)$ this maximum state vanishes [see Eq. (18c) and Ref. (23)].

[29] This is shown for $SU(n)$ by Baird and Biedenharn (Ref. 16). The latter authors' phase convention is $\langle\gamma\,|L^{(k,k+1)}|\,\beta\rangle \geq 0, k = 1, \cdots, n - 1$, a natural choice for the Gel'fand states, but one which does not reduce to the phase convention of de Swart (Ref. 8) for $SU(3)$, or the convention of C. L. Cook and G. Murtaza, Nuovo Cimento **39**, 531 (1965), for $SU(6)$.

Internal Multiplicity Structure and Clebsch-Gordan Series for the Exceptional Group $G(2)$

D. RADHAKRISHNAN AND T. S. SANTHANAM
MATSCIENCE, The Institute of Mathematical Sciences, Madras 20, India

(Received 10 April 1967)

An explicit algebraic formula is obtained for the multiplicity $\bar{M}(\gamma)$ of a vector γ belonging to the fundamental domain of the group $G(2)$. Using this, the internal multiplicity $M^m(m')$ of a weight m' of the irreducible representation $D(m)$ with the highest weight m is calculated through Kostant's formula for the dominant weights. The Clebsch–Gordan decomposition of the direct product of two irreducible representations is then obtained.

I. INTRODUCTION

It is well known that the group $G(2)$, which is a subgroup of $O(7)$, has been used extensively in nuclear physics[1] and in elementary particle physics[2] for classifying levels and for studying interactions between particles. It is desirable, therefore, that the Racah algebra of $G(2)$ be developed as in the familiar theory of angular momentum. The problem of finding the invariants has been solved.[3] Any irreducible representation (IR) is specified by the eigenvalues of the Casimir operators, or equivalently, by the components of the highest weight.

The next problem is the determination of the internal and external multiplicity structures[4] of the IR's of the group. According to Biedenharn's theorem,[5] the external multiplicity of an IR D'', occurring in the direct product of two IR's, D and D', is closely connected to the internal multiplicity of the weights in D or D'. Though the internal multiplicity structure is known through Kostant's formula,[6] practical computations with it are very tedious. It turns out that it is sufficient to know the multiplicity structure of $1/\Delta$.[7] Knowing this, the multiplicity $M^m(m')$ of a weight m' contained in an IR with highest weight m can be calculated.[8]

Recently, an algebraic method of getting $M^m(m')$ has been worked out[8] for the case of $SU(3)$. In the present paper we have obtained an expression for the internal multiplicity $M^m(m')$ for the group $G(2)$. The problem is more complicated in view of the fact that there are six negative roots and two (negative) primitive roots.

II. THE GROUP $G(2)$

The root diagram can be conveniently regarded as consisting of all vectors of the form $e_i - e_j$ and $e_i - 2e_j + e_k = (i, j, k = 1, 2, 3)$, which all belong to the hyperplane

$$\sum_{i=1}^{3} x_i = 0.$$

The negative primitive roots are

$$\beta_1 = (0, -1, 1) = e_3 - e_2,$$
$$\beta_2 = (-1, 2, -1) = -e_1 + 2e_2 - e_3.$$

The weight space is three dimensional with a subsidiary condition that

$$\sum_{i=1}^{3} m_i = 0,$$

where the m_i's are the components of the weight m. Using the theorem that $2(m, \alpha)/(\alpha, \alpha) =$ integer (where m is a weight and α is a root), it is clear that the components of m are integers.

Let us now discuss the Weyl group. Reflecting the weight (m_1, m_2, m_3) in the plane perpendicular to $e_i - e_j$, we see that $m_i \leftrightarrow m_j$, i.e., the components of m are permuted. Next consider the reflection in the plane perpendicular to $e_i - 2e_j + e_k$. It can be seen that the effect of this is to permute the components of m with a *total* change of sign. Thus, we have considered all possible reflections perpendicular to the roots and have seen that they permute the components of m or permute the components of m with an overall change in sign. The Weyl group is, therefore, of order 12. From these results, it follows that if $m = (m_1, m_2, m_3)$ is to be a dominant weight, then

$$\text{(a)} \quad m_1 \geq m_2 \geq m_3,$$
$$\text{(b)} \quad m_1 \geq 0, m_2 \leq 0, m_3 \leq 0.$$

---

[1] G. Racah, Phys. Rev. 76, 1352 (1949).
[2] R. E. Behrends, J. Dreitlein, C. Fronsdal, and W. Lee, Rev. Mod. Phys. 34, 1 (1962).
[3] B. Gruber and L. O'Raifeartaigh, J. Math. Phys. 5, 1796 (1964).
[4] We use the terminology introduced by A. J. Macfarlane, L. O'Raifeartaigh, and P. S. Rao, J. Math. Phys. 8, 536 (1967).
[5] L. C. Biedenharn, Phys. Letters 3, 254 (1963); G. E. Baird and L. C. Biedenharn, J. Math. Phys. 5, 1730 (1964).
[6] N. Jacobson, *Lie Algebras* (Interscience Publishers, Inc., New York, 1962), p. 261.
[7] J. P. Antoine and D. Speiser, J. Math. Phys. 5, 1226 (1964); B. Gruber, *ibid.* 7, 1797 (1966).
[8] B. Gruber and T. S. Santhanam, Nuovo Cimento 45A, 1046 (1966). A similar formula has been worked out for $SU(4)$ by B. Gruber, *ibid.* 48A, 23 (1967).

Proof: Assume (a) is not true, i.e., $m_r < m_{r+1}$ $(r = 1, 2)$. Applying such a Weyl reflection to m which exchanges m_r and m_{r+1}, we get a weight m' such that the first nonvanishing component is positive, thus leading to m' being higher than m. Hence $m_r \geq m_{r+1}$, which proves (a).

To prove (b), we note that condition (a), together with

$$\sum_{i=1}^{3} m_i = 0,$$

leads immediately to $m_1 \geq 0$ and $m_3 \leq 0$. We need to prove only that $m_2 \leq 0$. Assume the contrary, i.e., $m_2 > 0$. Applying such a reflection, which gives a weight m' with $-m_3$ as its first component, so that $m' - m$ has as its first component m_2 which is positive, we are lead to a contradiction. Hence, $m_2 \leq 0$.

III. MULTIPLICITY STRUCTURE $\bar{M}(k_1, k_2)$

In order to find the multiplicity of the dominant weights, let us first calculate the multiplicities $\bar{M}$ of the vectors in $1/\Delta$ using the expression[7]

$$\frac{1}{\Delta} = \sum_{a_1=0}^{\infty} \cdots \sum_{a_n=0}^{\infty} \exp i\left(\sum_{j=1}^{n} a_j\beta_j - R_0, \varphi\right), \tag{2}$$

where the a_i's are nonnegative integers, the β_j's are all the negative roots, and R_0 is half the sum of all positive roots. The multiplicity $\bar{M}$ of a particular vector γ of $1/\Delta$ (which belongs to the fundamental domain of a group of rank l),

$$\gamma = k_1\beta_1 + \cdots + k_l\beta_l - R_0, \tag{3}$$

where $(\beta_1, \cdots \beta_l)$ are the negative primitive roots $(l \leq n)$ and $(k_1 \cdots k_l)$ are nonnegative integers, is then given by the number of ways γ can be written as a sum over all the negative roots:

$$\gamma = \sum_{i=1}^{n} a_i\beta_i - R_0. \tag{4}$$

The multiplicity of the dominant weight m', $M^m(m')$ can then be obtained from[4]

$$\begin{aligned} M^m(m') &= \sum_{S\in W} \delta_S \bar{M}(m' - S(m + R_0)) \\ &= \sum_{S\in W} \delta_S \bar{M}(k_1^S, k_2^S), \end{aligned} \tag{5}$$

where the summation extends over the elements of the Weyl group W and $\delta_S = \pm 1$ according to whether S is an even or odd reflection, respectively. Equation (5) is Kostant's formula[6] for the dominant weights.

The problem of obtaining $\bar{M}(k_1, k_2)$ for $G(2)$ then reduces to finding the number of ways $(k_1\beta_1 + k_2\beta_2)$ can be expressed as $(a_1\beta_1 + \cdots + a_6\beta_6)$ for given k_1 and k_2, i.e.,

$$\begin{aligned} k_1\beta_1 + k_2\beta_2 = a_1\beta_1 + a_2\beta_2 + a_3(\beta_1 + \beta_2) \\ + a_4(2\beta_1 + \beta_2) + a_5(3\beta_1 + \beta_2) \\ + a_6(3\beta_1 + 2\beta_2), \end{aligned} \tag{6}$$

so that

$$\begin{aligned} k_1 &= a_1 + a_3 + 2a_4 + 3a_5 + 3a_6, \\ k_2 &= a_2 + a_3 + a_4 + a_5 + 2a_6. \end{aligned} \tag{7}$$

We have to find all the possible values allowed for $(a_1, \cdots, a_6)$ for given (k_1, k_2). These equations are known as Diophantine equations,[9] and we have solved them using the theory of partitions. One finds

$$\begin{aligned} &\bar{M}(k_1, k_2)_{k_1 \leq k_2} \\ &= (1 + k_1) + \sum_{i=0} \left[\frac{k_1 - i}{2}\right] \\ &\quad + \sum_{i,j=0} \left[\frac{k_1 - i - 2j}{3}\right] + \sum_{i,j,k=0} \left[\frac{k_1 - i - 2j - 3k}{3}\right] \\ &= (1 + k_1) + \left.\begin{matrix} \dfrac{k_1^2 - 1}{4} & (k_1 \text{ odd) or} \\ k_1^2/4 & (k_1 \text{ even}) \end{matrix}\right\}_{k_1 \geq 2} \\ &\quad + \tfrac{1}{2}\{(k_1 - 1)(k_1 - 4) + 4\}_{k_1 \geq 3} \\ &\quad + \bar{M}(k_1 - 3, k_2). \end{aligned} \tag{8}$$

We use the square bracket to denote the integral part of the expression.

$$\begin{aligned} &\bar{M}(k_1, k_2)_{k_1 \geq 3k_2} \\ &= (1 + k_2) + \sum_{i} (k_2 - i) \\ &\quad + \sum_{i,j} (k_2 - i - j) + \sum_{i,j} \left[\frac{k_2 - i - j - k}{2}\right] \\ &= \tfrac{1}{48}(k_2 + 2)(k_2^3 + 10k_2^2 + 30k_2 + 24), \quad \text{if } k_2 \text{ is even;} \end{aligned}$$

or

$$\tfrac{1}{48}(k_2 + 1)(k_2^3 + 11k_2^2 + 39k_2 + 45) \quad \text{if } k_2 \text{ is odd.} \tag{9}$$

[9] P. A. MacMahon, *Combinatory Analysis* (Chelsea Publishing Company, New York, 1960), Vol. II, Sec. VIII. The number of solutions of the Diophantine equations (7) can be given by the method of generating series. Now Eq. (7) can be written as a matrix equation

$$(a_1, \cdots, a_6)C = (k_1, k_2),$$

where C is a (6×2) matrix. The number of solutions of Eq. (7) is then obtained as the coefficient of $x_1^{k_1}x_2^{k_2}$ of the generating function

$$f(x_1, x_2) = \prod_{i=1}^{6} (1 - x_1^{c_{i1}}x_2^{c_{i2}})^{-1},$$

where the C_{ij} are the elements of the matrix C. We are grateful to P. K. Menon for drawing our attention to this fact.

$$\bar{M}(k_1, k_2)_{2k_2 \le k_1 < 3k_2}$$
$$= (1 + k_2) + \sum_i (k_2 - i) + \sum_{i+j+1 \le k_2} \left[\frac{k_1 - i - 2j}{3}\right] + \sum_{i+2j+3k+3 \le k_1} \left[\frac{k_2 - i - j - k}{2}\right]$$
$$= \tfrac{1}{2}(k_2 + 1)(k_2 + 2) + \tfrac{1}{2}\{(k_1 - 1)(k_1 - 4) + 4\}_{k_1 \ge 3} - \tfrac{1}{2}\{(k_1 - 2k_2)(k_1 - 2k_2 - 3) + 4\} \quad \text{for} \quad (k_1 - 2k_2) \ge 2,$$
$$+ \tfrac{1}{48}k_2\{(k_2 - 2)^3 + 10(k_2 - 2)^2 + 30(k_2 - 2) + 24\} \quad \text{for even} \quad k_2 \ge 2,$$
$$+ \tfrac{1}{48}(k_2 - 1)\{(k_2 - 2)^3 + 11(k_2 - 2)^2 + 39(k_2 - 2) + 45\} \quad \text{for odd} \quad k_2 \ge 3,$$
$$- \tfrac{1}{48}\mu\{(\mu - 2)^3 + 10(\mu - 2)^2 + 30(\mu - 2) + 24\} \quad \text{for even} \quad \mu \ge 2,$$
$$- \tfrac{1}{48}(\mu - 1)\{(\mu - 2)^3 + 11(\mu - 2)^2 + 39(\mu - 2) + 45\} \quad \text{for odd} \quad \mu \ge 3, \tag{10}$$

where
$$\mu = k_2 - 1 - [\tfrac{1}{3}(k_1 - 3)].$$

$$\bar{M}(k_1, k_2)_{3k_2 \le 2k_1 < 4k_2}$$
$$= (1 + k_2) + \sum_{i+1 \le k_2} \left[\frac{k_1 - i}{2}\right] + \sum_{i+j+1 \le k_2} \left[\frac{k_1 - i - 2j}{3}\right] + \sum_{i+2j+3k+3 \le k_1} \left[\frac{k_2 - i - j - k}{2}\right]$$
$$= (1 + k_2) + \frac{k_1^2 - 1}{4} \quad \text{for odd} \quad k_1 \ge 3,$$
$$+ \frac{k_1^2}{4} \quad \text{for even} \quad k_1 \ge 2,$$
$$- \left\{\frac{(k_1 - k_2)^2 - 1}{4}\right\} \quad \text{for odd} \quad (k_1 - k_2) \ge 3,$$
$$- \frac{(k_1 - k_2)^2}{4} \quad \text{for even} \quad (k_1 - k_2) \ge 2,$$
$$+ \tfrac{1}{2}\{(k_1 - 1)(k_1 - 4) + 4\} \quad \text{for} \quad k_1 \ge 3,$$
$$- \tfrac{1}{2}\{(k_1 - 2k_2)(k_1 - 2k_2 - 3) + 4\} \quad \text{for} \quad (k_1 - 2k_2) \ge 2,$$
$$+ \tfrac{1}{48}k_2\{(k_2 - 2)^3 + 10(k_2 - 2)^2 + 30(k_2 - 2) + 24\} \quad \text{for even} \quad k_2 \ge 2,$$
$$+ \tfrac{1}{48}(k_2 - 1)\{(k_2 - 2)^3 + 11(k_2 - 2)^2 + 39(k_2 - 2) + 45\} \quad \text{for odd} \quad k_2 \ge 3,$$
$$- \tfrac{1}{48}\mu\{(\mu - 2)^3 + 10(\mu - 2)^2 + 30(\mu - 2) + 24\} \quad \text{for even} \quad \mu \ge 2,$$
$$- \tfrac{1}{48}(\mu - 1)\{(\mu - 2)^3 + 11(\mu - 2)^2 + 39(\mu - 2) + 45\} \quad \text{for odd} \quad \mu \ge 3. \tag{11}$$

$$\bar{M}(k_1, k_2)_{2k_2 < 2k_1 < 3k_2}$$
$$= (1 + k_2) + \sum_{i+1 \le k_2} \left[\frac{k_1 - i}{2}\right] + \sum_{i+j+1 \le k_2} \left[\frac{k_1 - i - 2j}{3}\right] + \sum_{i+j+k+1 \le k_2} \left[\frac{k_1 - i - 2j - 3k}{3}\right]$$
$$= (1 + k_2) + \frac{k_1^2 - 1}{4} \quad \text{for odd} \quad k_1 \ge 3,$$
$$+ \frac{k_1^2}{4} \quad \text{for even} \quad k_1 \ge 2,$$
$$- \left\{\frac{(k_1 - k_2)^2 - 1}{4}\right\} \quad \text{for odd} \quad (k_1 - k_2) \ge 3,$$
$$- \left\{\frac{(k_1 - k_2)^2}{4}\right\} \quad \text{for even} \quad (k_1 - k_2) \ge 2,$$
$$+ \tfrac{1}{2}\{(k_1 - 1)(k_1 - 4) + 4\} \quad \text{for} \quad k_1 \ge 3,$$
$$- \tfrac{1}{2}\{(k_1 - 2k_2)(k_1 - 2k_2 - 3) + 4\} \quad \text{for} \quad (k_1 - 2k_2) \ge 2,$$
$$+ \bar{M}(k_1 - 3, k_2') \quad \text{with} \quad k_2' \ge k_1 - 3,$$
$$- \bar{M}(k_1 - k_2 - 2, k_2'') \quad \text{with} \quad k_2'' \ge (k_1 - k_2 - 2). \tag{12}$$

Equation (8) is a difference equation and can be solved for each modulus 3 of k_1. However, Eq. (8) itself is sufficient to determine $\bar{M}(k_1, k_2)(k_1 \le k_2)$ straightaway.

IV. MULTIPLICITY STRUCTURE $M^m(m')$

The multiplicity structure $M^m(m')$ is then given by Eq. (5). This is the number of ways a weight $m'(= m + k_1\beta_1 + k_2\beta_2)$ can be expressed as

$$m' = -R_0 + \sum_{i=1}^{6} a_i\beta_i + S(m + R_0), \qquad R_0 = (3, -1, -2). \tag{13}$$

We notice that when m' is dominant, only five Weyl reflections contribute to Eq. (5), the others leading necessarily to *negative* integer coefficients (k_1^S, k_2^S). From Eq. (5) we obtain

$$M^m(m') = \bar{M}\{(m_3' - m_1') + (m_1 - m_3); m_1 - m_1'\}$$
$$- \bar{M}\{(m_3' - m_1') + (m_1 - m_2) - 1; m_1 - m_1'\}$$
$$- \bar{M}\{(m_3' - m_1') + (m_1 - m_3); -(m_1' + m_3 + 1)\}$$
$$+ \bar{M}\{(m_3' - m_1') + (m_2 - m_3) - 4; -(m_1' + m_3 + 1)\}$$
$$+ \bar{M}\{(m_3' - m_1') + (m_1 - m_2) - 1; -(m_1' + m_2 + 2)\}. \tag{14}$$

Equation (14) along with Eqs. (8)–(12) give $M^m(m')$ for any dominant weight m'. The multiplicity of any other weight can be found by using the Weyl reflections

In Eqs. (8)–(12), the intervals for k_1 and k_2 depend sensitively on the coefficients of the a's in the Diophantine equations (7).

V. EXTERNAL MULTIPLICITY STRUCTURE

It is well known from the work of Biedenharn[5] that if $D(\Lambda)$ and $D'(\Lambda')$ are two IR's of a group L with Λ and Λ' as their highest weights, respectively, and if D' dominates[10] D, then the product $D' \times D$ contains IR's for which $(\Lambda' + m)$ are highest weights, where m stands for all weights contained in D. The multiplicity of the representation $(\Lambda' + m)$ in the reduction of $D' \times D$ is the same as the internal multiplicity of the weight m in the representation D. The conditions for D' to dominate D for $G(2)$ are[4]: $\lambda_1' \geq 2\lambda_1 + 3\lambda_2$, and $\lambda_2' \geq \lambda_1 + 2\lambda_2$, where (λ_1', λ_2') and (λ_1, λ_2) are the components of Λ' and Λ in the familiar two-component notation. More explicitly, Biedenharn's theorem can be stated in terms of characters:

$$\chi^{D\times D'}(\phi) = \sum_m \gamma_m \chi^{(\Lambda'+m)}(\phi). \tag{15}$$

The assumption that D' dominates D is needed to make $(\Lambda' + m)$ satisfy the conditions for it to be dominant so that it can be the highest weight of some representation in the reduction. The important point is that the representation with $(\Lambda' + m)$ as highest weight occurs γ_m times, where γ_m is the internal multiplicity of m in $D(\Lambda)$. γ_m can be immediately computed for any m in $D(\Lambda)$ using our results in Sec. IV. Thus, knowing $M^m(m')$ and Eq. (5), the Clebsch–Gordan reduction of the product of two IR's can be written down immediately. We give an example in the Appendix.

ACKNOWLEDGMENTS

The authors are grateful to Professor Alladi Ramakrishnan for his interest in this work and for his kind encouragement. They are also thankful to Dr. B. Gruber, Dr. D. Gaier, Dr. K. R. Unni, and Sri. M. R. Subramanya for very useful conversations.

APPENDIX

We give a few examples of multiplicities of some weights using the results obtained by us.

Consider the IR $D^7(1, 0)$, defined in the conventional $D^N(\lambda_1, \lambda_2)$ notation, where the highest weight (λ_1, λ_2) is given as λ_1 times one fundamental weight and λ_2 times the other. The connection with the three-component form is given by

$$m_1 = \lambda_1 + 2\lambda_2,$$
$$m_2 = -\lambda_2,$$
$$m_3 = -(\lambda_1 + \lambda_2).$$

We calculate the internal multiplicity of the dominant weight (0, 0). From Eq. (14) we find that

$$M^{(1,0)}(0, 0) = \bar{M}(2, 1) - \bar{M}(0, 1) - \bar{M}(2, 0).$$

Now using Eqs. (8)–(12), we find that

$$\bar{M}(2, 1) = 3, \quad \bar{M}(0, 1) = 1, \quad \bar{M}(2, 0) = 1,$$

so that

$$M^{(1,0)}(0, 0) = 1.$$

Similarly, for the internal multiplicity of the dominant weight (0, 0) in the representation $D^{14}(0, 1)$, we get

$$M^{(0,1)}(0, 0) = \bar{M}(3, 2) - \bar{M}(2, 2) - \bar{M}(3, 0)$$
$$= 7 - 4 - 1 = 2.$$

Let us now consider the direct product $D^{14}(0, 1) \times D^{1547}(3, 2)$. It can be seen that $D^{1547}(3, 2)$ dominates $D^{14}(0, 1)$. The various weights of $D^{14}(0, 1)$ are

$$\begin{array}{ccccc} (0, 1), & (3, -1), & (1, 0), & (-1, 1), & (2, -1), \\ (-3, 2), & (3, -2), & (-2, 1), & (1, -1), & (-1, 0), \\ (-3, 1), & (0, -1), & (0, 0), & (0, 0). & \end{array}$$

Using Biedenharn's theorem, Eq. (15), we see that

$$\begin{aligned} D^{14}(0, 1) \times D^{1547}(3, 2) &= D^{4096}(3, 3) + D^{3003}(6, 1) + D^{2926}(4, 2) \\ &\quad + D^{2079}(2, 3) + D^{1728}(5, 1) + D^{748}(0, 4) \\ &\quad + D^{714}(6, 0) + D^{896}(1, 3) + D^{729}(2, 2) \\ &\quad + D^{924}(4, 1) + D^{273}(0, 3) + D^{448}(3, 1) \\ &\quad + 2. D^{1547}(3, 2). \end{aligned}$$

It should be noted that the occurrence of $D^{1547}(3, 2)$ twice in the above reduction is due precisely to the appearance of the weight (0, 0) twice in $D^{14}(0, 1)$.

[10] See Ref. 4 for all details about the conditions for D' to dominate D. In this paper, a complete list of references to earlier literature may be found.

Unitary Representations of the Group $O(2, 1)$ in an $O(1, 1)$ Basis*

N. MUKUNDA†
Physics Department, Syracuse University, Syracuse, New York

(Received 6 April 1967)

We consider the unitary irreducible representations of the group $SO(2, 1)$, belonging to the continuous and the discrete classes. We cast them into a form in which the noncompact generator of an $O(1, 1)$ subgroup is diagonal. We examine some properties of the remaining generators in this basis. We recover the known result that the spectrum of the noncompact generator covers the real line twice for representations of the continuous class, and once for those of the discrete class.

INTRODUCTION

RECENTLY, there has been considerable interest in the possible uses of noncompact Lie groups for the description of physical systems. For example, certain groups have been suggested as noninvariance groups or spectrum-generating groups for simple quantum-mechanical systems and for elementary particles.[1] Generally for the purposes of physics one has to deal with unitary representations of Lie groups, and for a noncompact Lie group it is well known that any nontrivial unitary irreducible representation (UIR) must be infinite dimensional. Further, for many practical purposes it is necessary to have these representations in quite explicit form, with a suitable basis being chosen in the Hilbert space of the representation, and the infinitesimal generators of the Lie group being specified, if possible, by means of their matrix elements in the chosen basis.

In dealing with semi-simple noncompact Lie groups and their UIR's, two different possibilities arise. For certain groups it happens that in every UIR, each finite-dimensional UIR of the maximal compact subgroup appears once or not at all. Examples of such noncompact groups are the pseudo-orthogonal groups $O(p, 1)$ in $(p + 1)$ real dimensions, and the pseudounitary groups $SU(n, 1)$ in $(n + 1)$ complex dimensions. In these cases, a basis for the representation space for a UIR of the noncompact group can be constructed by taking an infinite sequence of distinct UIR's of the maximal compact subgroup, and the basis vectors are labelled completely by the Casimir invariants and internal "magnetic" quantum numbers of this subgroup. All of these labels are discrete variables, and the well-known algebraic techniques involving Clebsch–Gordan coefficients, the Wigner–Eckart theorem and reduced matrix elements with respect to the maximal compact subgroup, can in principle be used to build up UIR's of the noncompact group. On the other hand, for certain other semisimple noncompact groups, the situation is generally not so straightforward. A given UIR of the whole group may contain a given finite-dimensional UIR of the maximal compact subgroup several times. Examples are the pseudo-orthogonal groups $O(p,q)$ for p, $q \geq 2$, the pseudounitary groups $SU(n, m)$ for n, $m \geq 2$, and the special linear groups $SL(n, R)$, $SL(n, C)$ for $n \geq 3$. In these cases, the Casimir invariants and internal "magnetic" quantum numbers of the maximal compact subgroup do not suffice to completely label the basis vectors of an UIR of the whole group, and one needs additional operators to distinguish the several occurrences of the same UIR of the maximal compact subgroup. [Even for such groups, of course, there may be special classes of UIR's, generally called degenerate UIR's, in which there is no multiplicity of occurrence of UIR's of the maximal compact subgroup.] In such a situation, the previously mentioned algebraic methods are enormously harder to apply.

These remarks suggest that one examine the UIR's of the second kind of noncompact semisimple groups by reducing the UIR's with respect to a noncompact subgroup, this subgroup being chosen to be "large enough" so that the multiplicity problem is (almost) removed.[2] For example, one can ask how the UIR's of the de Sitter group $O(3, 2)$ are constructed by putting together UIR's of the $O(3, 1)$ subgroup, rather than of the maximal compact subgroup $O(3) \otimes O(2)$. Of course, it is clear that this immediately

* Supported in part by the U.S. Atomic Energy Commission.
† Present address: Tata Institute of Fundamental Research, Bombay, India.
[1] A partial list of references: A. O. Barut, Phys. Rev. **135**, B839 (1964); Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965); T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965); N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965); C. Fronsdal, in *Proceedings of the Third Coral Gables Conference* (W. H. Freeman & Company, San Francisco, California, 1966); E. C. G. Sudarshan, *ibid.*; Y. Nambu, "Relativistic Wave Equations for Particles with Internal Structure and Mass Spectrum," University of Chicago Preprint (1966).

[2] This kind of reduction of representations has been attempted in several recent papers. See, for example, J. Niederle, ICTP Preprint IC/66/99, Trieste (1966).

raises many problems. For example, one will have to deal in general with direct integrals, rather than direct discrete sums, of UIR's of the noncompact subgroup, and these latter UIR's will also be infinite dimensional. Further, in such a basis, the infinitesimal generators of the whole Lie group may have to be treated with greater care, and the use of matrix elements in a simple-minded way may not be possible.

In this paper, we consider the simplest problem of this genre. We examine the UIR's of the pseudo-orthogonal group in three dimensions, $O(2, 1)$, cast them into a form in which the reduction with respect to the noncompact subgroup $O(1, 1)$ is apparent, and then examine the nature of the infinitesimal generators in this basis.[3] As is well known, the Lie algebra of $O(2, 1)$ is the same as the Lie algebra of the group $SU(1, 1)$ of pseudounitary unimodular matrices in two dimensions. Of course, in this example, the maximal compact subgroup $O(2)$ is "large enough" so that there is no multiplicity problem in the reduction of UIR's of $O(2, 1)$ [or $SU(1, 1)$] with respect to $O(2)$. This is also the case when one considers the group $O(3, 1)$. Nevertheless, these examples are interesting in themselves, and they may serve to point out some features to be expected when one treats more complicated cases like, say, the reduction of UIR's of $O(3, 2)$ with respect to $O(3, 1)$.

We outline briefly the contents of the paper. In Sec. I, we recapitulate some familiar facts and properties of the group $SU(1, 1)$ and of its Lie algebra. In Sec. II, we describe briefly the different kinds of UIR's of $SU(1, 1)$, restricting attention to single valued UIR's of $SU(1, 1)$. This corresponds to restricting oneself to single- and double-valued UIR's of $O(2, 1)$, since, as is well known, there is a two-to-one homomorphism from $SU(1, 1)$ to $O(2, 1)$. These UIR's will be written, as usual, in a form which is already reduced with respect to the maximal compact subgroup of $SU(1, 1)$. The material in the first two sections is collected together only in order to make this paper reasonably self-contained.[4] Sections III and IV are devoted to examining two classes of UIR's of $O(2, 1)$, namely the continuous (nonexceptional) and the discrete classes. We exhibit them in a form suited to the reduction with respect to a noncompact subgroup $O(1, 1)$ of $O(2, 1)$. In both cases, for simplicity, we restrict ourselves to single-valued UIR's of $O(2, 1)$. There exists also a class of UIR's of $SO(2, 1)$ called the exceptional class. The treatment of these UIR's will be taken up in a later publication. We also discuss in a subsequent paper the reduction of UIR's of $O(3, 1)$ with respect to its noncompact subgroup $O(2, 1)$.

I. RÉSUMÉ OF THE GROUP $SU(1, 1)$

It is well known that the Lie algebras of $SU(1, 1)$ and of $O(2, 1)$ are the same, and that there is a two-to-one homomorphism from $SU(1, 1)$ to $O(2, 1)$. Though we will later analyze only certain single valued UIR's of $O(2, 1)$, we shall describe here the structure and properties of $SU(1, 1)$ since the corresponding matrices are easier to deal with.

The group $SU(1, 1)$ is the group of all two-dimensional complex pseudounitary matrices of unit determinant. In other words, it is the group of all complex unimodular linear transformations on two complex variables x_1, x_2 leaving the quadratic form

$$|x_1|^2 - |x_2|^2$$

invariant. A general element g of $SU(1, 1)$ corresponds to a matrix

$$g \to \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1 \tag{1.1}$$

with α, β being complex numbers and the bars denoting complex conjugation. Distinct matrices correspond to distinct elements of the abstract group. The parameters α, β, obeying $|\alpha|^2 - |\beta|^2 = 1$, are equivalent to three real numbers so that g is described by three real parameters. Analogous to the Euler angle characterization of the three-dimensional orthogonal rotation group, every matrix of the form (1.1) can be expressed as a product of three factors in the following way:

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{pmatrix} \begin{pmatrix} \cosh \zeta/2 & \sinh \zeta/2 \\ \sinh \zeta/2 & \cosh \zeta/2 \end{pmatrix} \times \begin{pmatrix} e^{i\mu'/2} & 0 \\ 0 & e^{-i\mu'/2} \end{pmatrix}. \tag{1.2}$$

Each factor in this product is itself an element of $SU(1, 1)$. The three real parameters μ, ζ, μ' are not uniquely determined by α and β. It is enough to say that, on allowing these parameters to vary over the ranges

$$-2\pi \le \mu, \quad \mu' < 2\pi, \quad 0 \le \zeta < \infty, \tag{1.3}$$

we do obtain all elements of the group $SU(1, 1)$; for $\zeta \neq 0$, every element is obtained twice, while for $\zeta = 0$, the only quantity that is relevant is the sum of the other two parameters, $\mu + \mu'$.

[3] The classic work on the unitary representations of $O(2, 1)$ is V. Bargmann, Ann. Math. **48**, 568 (1947). Some recent papers concerned with this group are: A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A287**, 532 (1965); A. Kihlberg, Arkiv Fysik **30**, 121 (1965); W. J. Holmann III and L. C. Biedenharn, Ann. Phys. **39**, 1 (1966).

[4] This material is taken from V. Bargmann, Ref. 3.

The first and third factors appearing on the right-hand side of Eq. (1.2) are elements of the maximal compact subgroup of $SU(1, 1)$; this subgroup consists of all elements corresponding to matrices of the form

$$\begin{pmatrix} e^{-i\mu/2} & 0 \\ 0 & e^{i\mu/2} \end{pmatrix}, \quad -2\pi \le \mu < 2\pi. \tag{1.4}$$

There is a two-to-one homomorphism between the elements of the type (1.4) of $SU(1, 1)$ and the elements of the maximal compact subgroup $O(2)$ of $O(2, 1)$. The second factor on the right-hand side of (1.2) is an element of a certain noncompact subgroup of $SU(1, 1)$; this subgroup consists of all elements corresponding to matrices of the form

$$\begin{pmatrix} \cosh \zeta/2 & \sinh \zeta/2 \\ \sinh \zeta/2 & \cosh \zeta/2 \end{pmatrix}, \quad -\infty < \zeta < \infty. \tag{1.5}$$

The elements of the form (1.5) are in one-to-one correspondence with the elements of a noncompact subgroup $O(1, 1)$ of the group $O(2, 1)$.

The Lie algebra of $SU(1, 1)$ [or of $O(2, 1)$] contains three linearly independent elements, which we denote by J_0, J_1, and J_2. They fulfill the following commutation relations:

$$-i[J_0, J_1] = J_2, \tag{1.6a}$$

$$-i[J_0, J_2] = -J_1, \tag{1.6b}$$

$$-i[J_1, J_2] = -J_0. \tag{1.6c}$$

In a unitary representation of $SU(1, 1)$ all the three operators J_0, J_1, and J_2 will be represented by self-adjoint linear operators. J_0 is the generator of the maximal compact subgroup of $SU(1, 1)$ while J_1 and J_2 are the so-called "noncompact" generators. In the language of Lorentz transformations in three dimensions, namely the group $O(2, 1)$, J_0 generates spatial rotations in a plane [the subgroup $O(2)$], while J_1 and J_2 are the generators of accelerations (pure velocity transformations) in the two independent directions. The transformations generated by J_1 form an $O(1, 1)$ subgroup of $O(2, 1)$, and similarly for J_2. The Casimir invariant of the Lie algebra of $SU(1, 1)$ is the quadratic operator Q defined by

$$Q = J_1^2 + J_2^2 - J_0^2. \tag{1.7}$$

Q commutes with J_0, J_1, and J_2, and in any UIR it is equal to a real multiple of the identity operator.

It is useful to identify the matrices corresponding to the generators J_0, J_1, and J_2 in the defining nonunitary representation of $SU(1, 1)$, [given by (1.1)], and the elements belonging to the one-parameter subgroups generated by them. We may choose

$$J_0 = \tfrac{1}{2}\sigma_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad J_1 = \tfrac{1}{2}i\sigma_2 = \tfrac{1}{2}i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_2 = -\tfrac{1}{2}i\sigma_1 = -\tfrac{1}{2}i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.8}$$

Let the elements of the one-parameter subgroups referred to above be denoted by $\exp(i\mu J_0)$, $\exp(i\nu J_1)$, and $\exp(i\zeta J_2)$, respectively. These elements correspond to the matrices

$$\exp(i\mu J_0) \to \begin{pmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{pmatrix}, \tag{1.9a}$$

$$\exp(i\nu J_1) \to \begin{pmatrix} \cosh \nu/2 & i \sinh \nu/2 \\ -i \sinh \nu/2 & \cosh \nu/2 \end{pmatrix}, \tag{1.9b}$$

$$\exp(i\zeta J_2) \to \begin{pmatrix} \cosh \zeta/2 & \sinh \zeta/2 \\ \sinh \zeta/2 & \cosh \zeta/2 \end{pmatrix}. \tag{1.9c}$$

The elements $\exp(i\mu J_0)$, $\exp(i\zeta J_2)$ belong to the two subgroups described earlier [Eqs. (1.4) and (1.5)], and according to Eq. (1.2) all elements of the group $SU(1,1)$ may be obtained by taking suitable products of such special elements.

II. UNITARY REPRESENTATIONS OF $SU(1, 1)$

The single-valued UIR's of $SU(1, 1)$ have been determined long ago by Bargmann,[5] and they fall into several distinct classes. Each UIR can be characterized by the value of the Casimir invariant Q, and the spectrum of eigenvalues of the element J_0 of the Lie algebra of $SU(1, 1)$. [The value of the Casimir invariant is not always enough to uniquely specify a UIR of $O(2, 1)$, in contrast to the case of $O(3)$.] The restriction to single-valued representations of $SU(1, 1)$ implies that the eigenvalues of J_0 are either integers or half-odd integers. Within a given UIR, the eigenvalues of J_0 differ from one another by integers. Denote the eigenvalues of Q by q and those of J_0 by m. The different classes of UIR's are the following:

(A) *Continuous class, integral case, nonexceptional interval:*

$$\tfrac{1}{4} \le q < \infty, \quad m = 0, \pm 1, \pm 2, \cdots, \text{ad inf.}$$

(B) *Continuous class, exceptional interval:*

$$0 < q < \tfrac{1}{4}, \quad m = 0, \pm 1, \pm 2, \cdots, \text{ad inf.}$$

(C) *Continuous class, half-integral case:*

$$\tfrac{1}{4} < q < \infty, \quad m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \cdots, \text{ad inf.}$$

(D) *Discrete class, positive m:*

$$q = k(1 - k), \quad k = \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots.$$

[5] V. Bargmann, Ref. 3.

For given k, we have $m = k, k+1, k+2, \cdots$, ad inf.

(E) *Discrete class, negative m:*

$$q = k(1-k), \quad k = \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots.$$

For given k, we have $m = -k, -k-1, -k-2, \cdots$, ad inf.

The UIR's of types (A) and (B) are together denoted by C_q^0, those of type (C) by $C_q^{\frac{1}{2}}$, and those of types (D) and (E) by D_k^+ and D_k^-, respectively. Representations with integral (half-odd integral) values of m are single (double) valued representations of $O(2,1)$.

The form of the generators J_0, J_1, J_2 in each of the classes of UIR's listed above can be given by introducing an orthonormal basis consisting of eigenvectors of the generator J_0. Let us denote the elements of such a basis by $|m\rangle$. Then we have

$$J_0 |m\rangle = m |m\rangle; \quad \langle m' \mid m\rangle = \delta_{m',m}. \tag{2.1}$$

The action of the generators J_1 and J_2 in this basis is given by the following equations:

$$\begin{aligned} J_1 |m\rangle &= \tfrac{1}{2}[q + m(m+1)]^{\frac{1}{2}} |m+1\rangle \\ &\quad + \tfrac{1}{2}[q + m(m-1)]^{\frac{1}{2}} |m-1\rangle, \\ J_2 |m\rangle &= -\frac{i}{2}[q + m(m+1)]^{\frac{1}{2}} |m+1\rangle \\ &\quad + \frac{i}{2}[q + m(m-1)]^{\frac{1}{2}} |m-1\rangle. \end{aligned} \tag{2.2}$$

The value of the parameter q, and the range of values of m, is appropriate to the particular UIR.

We make a few remarks concerning the form of (2.2). From (1.6a) and (1.6b) it is clear that the operators J_1, J_2 transform as the components of a real two-dimensional vector under the $O(2)$ rotations generated by J_0. It is this tensor character of J_1, J_2 with respect to $O(2)$ that results in the selection rules $\Delta m = \pm 1$ for the matrix elements of J_1, J_2 in a basis made up of eigenvectors of J_0. In fact, the non-Hermitian operators $J_\pm = J_1 \pm iJ_2$ act as raising and lowering operators with respect to the eigenvalues of J_0, as is clear from the following commutation relations:

$$[J_0, J_+] = J_+, \quad [J_0, J_-] = -J_-. \tag{2.3}$$

We are interested in examining the UIR's of $SU(1,1)$ in a basis in which the generator J_2 is diagonal. In this case, (1.6b) and (1.6c) show that under the $O(1,1)$ transformations generated by J_2, J_0, and J_1 go over into linear combinations of themselves according to a nonunitary reducible matrix representation of $O(1,1)$:

$$\begin{aligned} \exp(i\zeta J_2)\colon\ J_0 &\to (\cosh \zeta) J_0 - (\sinh \zeta) J_1, \\ J_1 &\to (-\sinh \zeta) J_0 + (\cosh \zeta) J_1. \end{aligned} \tag{2.4}$$

We can form combinations of J_0, J_1 which have commutation relations with J_2 analogous to (2.3). These combinations are $K_\pm = J_0 \pm J_1$, and we have

$$[J_2, K_+] = iK_+, \quad [J_2, K_-] = -iK_-. \tag{2.5}$$

In contrast to the earlier case, we see that the combinations $J_0 \pm J_1$ are *real* linear combinations of the generators, and that if we were to naively interpret $J_0 \pm J_1$ as "raising" and "lowering" operators with respect to the eigenvalues of J_2, then under the application of these operators, the eigenvalue of J_2 gets shifted by $\pm i$. This of course does not make sense in a unitary representation of $SU(1,1)$ because in that case J_2 is a self-adjoint operator with a purely real spectrum.

For the rest of the paper, we restrict ourselves to UIR's of the type (A), namely C_q^0 with $q \geq \frac{1}{4}$, and some of those of type (D), namely D_k^+ with $k = 1, 2, 3, \cdots$. The object will be to examine these UIR's in a basis where J_2 is diagonal, and attempt to interpret the commutation relations (2.5) in this basis. The UIR's C_q^0, $q \geq \frac{1}{4}$, are treated in Sec. III, and the UIR's D_k^+, for integral $k \geq 1$, in Sec. IV.

III. REPRESENTATIONS OF THE CONTINUOUS CLASS

For UIR's of the class C_q^0, $\frac{1}{4} \leq q < \infty$, we may write

$$q = \tfrac{1}{4} + s^2, \quad 0 \leq s < \infty. \tag{3.1}$$

These UIR's can be realized by unitary transformations in a Hilbert space H of (Lebesgue) square-integrable functions (L^2 functions) on the unit circle.[6] Elements of H correspond to functions $f(\varphi)$ of the real variable φ varying in the range $0 \leq \varphi < 2\pi$. (This correspondence is of course only up to sets of measure zero; however, we will not state this repeatedly.) In the usual terminology of quantum mechanics, we think of $f(\varphi)$ as the "wavefunction" representing the abstract vector f in the "φ basis." The inner product of an element f with an element h, and the norm of an element f, denoted, respectively, by (f, h) and $\|f\|$, are given by

$$\begin{aligned} (f, h) &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \overline{f(\varphi)} h(\varphi), \\ \|f\| &= (f, f)^{\frac{1}{2}} < \infty. \end{aligned} \tag{3.2}$$

Let g be an element of $SU(1,1)$, corresponding to a

[6] V. Bargmann, Ref. 3, p. 613.

matrix specified by the parameters α, β as in (1.1). To g corresponds a unitary operator $U(g)$, and the wavefunction of the vector $U(g)f$ is given in terms of that of f by

$$\left.\begin{aligned} [U(g)f](\varphi) &= |\bar{\alpha} - \beta e^{i\varphi}|^{-1-2is} f(\psi_g(\varphi)), \\ e^{i\psi_g(\varphi)} &= \frac{\alpha e^{i\varphi} - \bar{\beta}}{\bar{\alpha} - \beta e^{i\varphi}}, \quad 0 \le \varphi, \quad \psi_g(\varphi) < 2\pi. \end{aligned}\right\} \tag{3.3}$$

The operators $U(g)$, for fixed real $s \ge 0$, give the UIR C_q^0. The forms of the generators J_0, J_1, J_2 are obtained by specializing the element g in (3.3) to the matrices of (1.9). One finds:

$$\begin{aligned} [J_0 f](\varphi) &= \frac{1}{i}\frac{df(\varphi)}{d\varphi}, \\ [J_1 f](\varphi) &= \frac{1}{i}\cos\varphi\,\frac{df(\varphi)}{d\varphi} + i(\tfrac{1}{2} + is)\sin\varphi f(\varphi), \\ [J_2 f](\varphi) &= \frac{1}{i}\sin\varphi\,\frac{df(\varphi)}{d\varphi} - i(\tfrac{1}{2} + is)\cos\varphi f(\varphi). \end{aligned} \tag{3.4}$$

Apart from m-dependent phase factors, the orthonormal basis states $|m\rangle$ of (2.1) correspond to the following functions:

$$|m\rangle \to e^{im\varphi}. \tag{3.5}$$

We would now like to convert this form of the UIR C_q^0 into a form where J_2 is diagonal. This is achieved by a change of variable as follows.[7] We define a real variable q as a function of φ:

$$\begin{aligned} e^{q} &= \tan \varphi/2: & 0 \le \varphi \le \pi, \\ e^{-q} &= \tan (\varphi - \pi)/2: & \pi \le \varphi \le 2\pi. \end{aligned} \tag{3.6}$$

As φ varies from 0 to π (upper half of the circumference of the unit circle), q varies continuously from $-\infty$ to $+\infty$, taking on each value in this range exactly once. As φ varies from π to 2π (lower half of the circumference of the unit circle), q goes continuously over the range $+\infty$ to $-\infty$, taking on each value once only. By this transformation, the circumference of the unit circle is mapped onto *two real lines*. The effect of this mapping on the functions $f(\varphi)$ must be specified next. If an element f in H is specified by the wavefunction $f(\varphi)$ in the φ basis, we associate with it a new wavefunction in the q basis. We define *two* functions $f_1(q)$, $f_2(q)$ in terms of $f(\varphi)$:

$$\begin{aligned} f_1(q) &= [\cosh q]^{-\frac{1}{2}-is} f(\varphi): & 0 \le \varphi \le \pi, \\ f_2(q) &= [\cosh q]^{-\frac{1}{2}-is} f(\varphi): & \pi \le \varphi \le 2\pi. \end{aligned} \tag{3.7}$$

[7] An alternative, more direct, method has been used by J. G. Kuriyan, Ph.D. thesis, Department of Physics, Syracuse University (1966). For details, see J. G. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, "The Theory of Master Analytic Representations of Compact and Noncompact Groups," Institute for Advanced Study, Princeton, New Jersey (1967) (to be published).

The relation between q and φ, for each range of φ, is as given in (3.6). Thus each wavefunction $f(\varphi)$ is replaced by two functions of q, one on each of the two real lines. The values of $f(\varphi)$ on the upper (lower) semicircle determine the function $f_1(q)[f_2(q)]$. Writing these as a two-component column vector, we may say that the correspondence between an element f in H and its wavefunction in the q basis is given by

$$f \in H \to \begin{pmatrix} f_1(q) \\ f_2(q) \end{pmatrix}, \quad -\infty < q < \infty. \tag{3.8}$$

It may be helpful to remark that the change of variable we have made from φ to q is of the expected kind. The compact generator is associated with rotations of a circle, while the noncompact one is associated with translations of a line. So we expect a trigonometric function of φ to be equal to a hyperbolic function of q.

The scalar product of two elements f, h can be expressed in terms of the new wavefunctions by combining (3.2), (3.6), and (3.7). We get

$$(f, h) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq[\overline{f_1(q)}h_1(q) + \overline{f_2(q)}h_2(q)]. \tag{3.9}$$

We see immediately that H has been expressed as the direct sum of two Hilbert spaces, H_1 and H_2, each consisting of all (Lebesgue) square-integrable functions on the entire real line. It is important to realize that in order to "recover" all elements of H, we have to consider all wavefunctions of the type (3.8), in which $f_1(q)$ and $f_2(q)$ are independently chosen square-integrable functions on the real line.

The next step is to express the generators J_0, J_1, J_2 as well as the unitary operators $U(g)$, in the new basis. Proceeding purely formally making use of (3.4), (3.6), and (3.7), we find:

$$\begin{aligned} [J_0 f]_1(q) &= \frac{1}{i}\cosh q\,\frac{df_1(q)}{dq} - i(\tfrac{1}{2} + is)\sinh q f_1(q), \\ [J_0 f]_2(q) &= \frac{-1}{i}\cosh q\,\frac{df_2(q)}{dq} + i(\tfrac{1}{2} + is)\sinh q f_2(q), \\ [J_1 f]_1(q) &= \frac{-1}{i}\sinh q\,\frac{df_1(q)}{dq} + i(\tfrac{1}{2} + is)\cosh q f_1(q), \\ [J_1 f]_2(q) &= \frac{1}{i}\sinh q\,\frac{df_2(q)}{dq} - i(\tfrac{1}{2} + is)\cosh q f_2(q), \\ [J_2 f]_r(q) &= \frac{1}{i}\frac{d}{dq}f_r(q), \quad r = 1, 2. \end{aligned} \tag{3.10}$$

It is simple to write these expressions in block form (remembering that elements in H are represented by two-component wavefunctions) and as differential operators acting directly on wavefunctions:

$$J_0 = \left[\frac{1}{i}\cosh q\,\frac{d}{dq} - i(\tfrac{1}{2} + is)\sinh q\right] \otimes \sigma,$$

$$J_1 = \left[i\sinh q\,\frac{d}{dq} + i(\tfrac{1}{2} + is)\cosh q\right] \otimes \sigma, \quad (3.11)$$

$$J_2 = \frac{1}{i}\frac{d}{dq} \otimes 1.$$

At first sight, (3.11) indicates that *all* the generators J_0, J_1, J_2 leave the subspaces H_1 and H_2, consisting, respectively, of elements with $f_2(q) = 0$ and $f_1(q) = 0$, invariant. However, this certainly cannot be the case, since we are dealing with an *irreducible* representation of the group $SU(1, 1)$. The point is that the linear differential operators (3.11) have been derived in a formal manner starting from (3.4); these differential operators by themselves do not determine the self-adjoint operators J_0, J_1, J_2, but must be supplemented by statements concerning the domains of the operators. It is the latter that show how the subspaces H_1 and H_2 are connected. We can see this in a somewhat simpler way, by directly expressing the effect of the unitary operator $U(g)$ on the wavefunctions $f_r(q)$. Let us consider the operator J_2 first. We specialize g in (3.3) to members of the one-parameter subgroup generated by J_2, as given in (1.9c). Combining this with (3.3), (3.6), and (3.7), we find

$$[U(\exp(i\zeta J_2))f]_r(q) = f_r(q + \zeta), \quad r = 1, 2. \quad (3.12)$$

This shows that the subgroup generated by J_2, and hence J_2 itself, leaves H_1 and H_2 invariant. Considering next the case of J_0, we have to express the functions

$$[U(\exp(i\mu J_0))f]_r(q)$$

in terms of $f_r(q)$. This can be done easily, but leads to rather cumbersome expressions. The essential point we wish to demonstrate, however, is that the subspaces H_1 and H_2 are not left invariant by $U(\exp(i\mu J_0))$, and this can be seen directly as follows. In the φ basis, we know from (1.9a) and (3.3) that

$$[U(\exp(i\mu J_0))f](\varphi) = f(\varphi + \mu), \quad (3.13)$$

namely $U(\exp(i\mu J_0))$ just produces a rotation of the unit circle on which $f(\varphi)$ is defined. However, this means that a portion of the upper semicircle goes over into a portion of the lower one, and vice versa. In terms of the two real lines on which f_1 and f_2 are defined, this means that part of one real line gets mapped onto part of the other, and vice versa. In other words, the function

$$[U(\exp(i\mu J_0))f]_1(q)$$

is given, for $-\infty < q \leq \ln\cot\frac{1}{2}\mu$, in terms of the function f_1, and for $\ln\cot\frac{1}{2}\mu \leq q < \infty$, in terms of the function f_2. In a similar way, one can see that the function

$$[U(\exp(i\mu J_0))f]_2(q)$$

is specified in terms of f_1 for a certain range of q, and in terms of f_2 over the rest. (We have assumed $0 \leq \mu \leq \pi$ for definiteness.) It is now clear that neither the unitary operators $U(\exp(i\mu J_0))$, nor the self-adjoint generator J_0, leaves either H_1 or H_2 invariant. The same can be seen to be true of the operators $U(\exp(i\nu J_1))$ and J_1. Thus, though the generators (3.10) appear to leave H_1 and H_2 invariant, they do not do so because of the nature of the domain of definition of the generators. For example, a vector f in the domain of J_0 has components f_1 and f_2, in H_1 and H_2, which are constrained by one another and cannot be chosen completely independently. These constraints involve $f_{1,2}(q)$ at $q = \pm\infty$, and reflect the continuity and differentiability of the associated function $f(\varphi)$.

We could now go ahead and express $U(g)$, for the most general element g in $SU(1, 1)$, by means of its action on the wavefunction $f_r(q)$. This may be achieved by a series of variable changes using (3.3), (3.6), and (3.7). The results are however quite unwieldy, and are not given here. We instead next consider the question of the spectrum of the generator J_2, and of the nature of the generators J_0 and J_1 in a basis where J_2 is diagonal.

The effect of J_2, as well as of $U(\exp(i\zeta J_2))$, on an element f with wavefunctions $f_r(q)$, has been given in (3.10) and (3.12). By carrying out a Fourier transformation with respect to q, in each of the subspaces H_1 and H_2, we pass to a basis in which J_2 is diagonal. (The process of taking the Fourier transform of a square-integrable function on the real line is, of course, a unitary operation.) We denote the Fourier transforms of $f_r(q)$ by $f'_r(p)$[8],

$$f'_r(p) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-ipq} f_r(q)\,dq, \quad r = 1, 2. \quad (3.14)$$

In this "p representation," a vector f is represented by the wavefunctions $f'_r(p)$. The scalar product (3.9) reads

$$(f, h) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp[\overline{f'_1(p)}h'_1(p) + \overline{f'_2(p)}h'_2(p)]. \quad (3.15)$$

[8] We have added a prime to the wavefunctions $f'_r(p)$, to distinguish them from the q-space wavefunctions $f_r(q)$, to which they are related by a Fourier transformation. These different functions are representatives of the same abstract vector f in different bases.

In this basis, J_2 is diagonal, since we have

$$[J_2f]_r'(p) = pf_r'(p),$$
$$[U(\exp(i\zeta J_2))f]_r'(p) = e^{i\zeta p}f_r'(p), \quad r = 1, 2. \tag{3.16}$$

We see explicitly that the spectrum of J_2 consists of the entire real line, and in fact every "eigenvalue" appears *twice*, corresponding to the two values of the index r. This is in accord with the statement of Bargmann,[9] and is characteristic of the UIR's C_q^0.

We consider next the generators J_0 and J_1 in this basis. Since the spectrum of J_2 is continuous, it has no normalizable eigenvectors. We can, however, introduce "ideal" eigenvectors, subjected to a delta-function normalization, analogous to momentum eigenstates in quantum mechanics. We can then say that the "basis" for H is made up of the ideal vectors

$$|p, r\rangle: \quad -\infty < p < +\infty, \quad r = 1, 2, \tag{3.17}$$

with the properties

$$\langle p', r' \mid p, r\rangle = \delta(p' - p)\delta_{r'r},$$
$$J_2 |p, r\rangle = p\,|p, r\rangle. \tag{3.18}$$

A normalizable vector is expanded in this basis by using its wavefunction as coefficients in the expansion

$$f \to |f\rangle = \sum_r \int_{-\infty}^{\infty} dp f_r'(p)\,|p, r\rangle. \tag{3.19}$$

We form the linear combinations $K_\pm = J_0 \pm J_1$ which appear as "raising" and "lowering" operators with respect to J_2 [cf. Eq. (2.5)]; in the q representation, we have

$$K_\pm = e^{\mp q}\left[\frac{1}{i}\frac{d}{dq} \pm i(\tfrac{1}{2} + is)\right] \otimes \sigma_3. \tag{3.20}$$

We would now like to interpret the commutation relations (2.5), and see how $K_\pm$ behave in the p representation (J_2 diagonal). It turns out that the essential feature in the understanding of (2.5) is already seen when we consider the simpler operators

$$K_\pm^{(0)} = e^{\mp q} = \lim_{s\to\infty} (K_\pm/-s),$$
$$J_2 = (1/i)\, d/dq. \tag{3.21}$$

They obey the commutation relations

$$[J_2, K_\pm^{(0)}] = \pm i K_\pm^{(0)},$$
$$[K_+^{(0)}, K_-^{(0)}] = 0. \tag{3.22}$$

In dealing with the generators J_2, $K_\pm^{(0)}$, we can drop the multiplicity label r, and restrict ourselves to the subspace H_1. The operators (3.21) generate the Poincaré group in one space and one time dimension. In that case, there is no degeneracy index r; however, in the case of interest to us, namely $SU(1, 1)$, the index r is essential. In the interest of simplicity, and of presenting the main points as clearly as possible, we first examine the operators (3.21).

Let us consider the operator $K_+^{(0)}$. In the q basis, a vector f in H with wavefunction $f(q)$ will lie in the domain $K_+^{(0)}$ if and only if

$$\|K_+^{(0)}f\|^2 = \int_{-\infty}^{\infty} e^{-2q}\,|f(q)|^2\, dq < \infty \tag{3.23}$$

in addition to

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(q)|^2\, dq < \infty. \tag{3.24}$$

Clearly, (3.23) imposes severe restrictions on $f(q)$ as $q \to -\infty$. If this vector f is also in the domain of J_2, we have

$$[J_2f](q) = (1/i)[df(q)/dq]. \tag{3.25}$$

Let us write

$$K_+^{(0)}f = h, \quad h(q) = e^{-q}f(q). \tag{3.26}$$

To see the behavior of $K_+^{(0)}$ when J_2 is diagonal, we must relate the Fourier transform of $h(q)$ to that of $f(q)$. From the fact that $h(q)$ is normalizable, we expect that the Fourier transform of $f(q)$ can be analytically continued into the lower half of the complex plane. By means of standard theorems on Fourier and Laplace transforms,[10] we find that if (3.23) and (3.24) both hold, then we can define a function ψ of a complex variable $z = p - i\alpha$ by

$$\psi(p - i\alpha) = (2\pi)^{-\frac{1}{2}} \underset{R\to\infty}{\text{l.i.m.}} \int_{-R}^{R} e^{-ipq-\alpha q}f(q)\, dq. \tag{3.27}$$

If $-1 \le -\alpha = \operatorname{Im} z \le 0$, the integral converges in the sense of the limit in the mean (l.i.m.) to the function $\psi(p - i\alpha)$. For $\alpha = 0$, $\psi(p)$ is just the Fourier transform of $f(q)$, while for $\alpha = 1$, $\psi(p - i)$ is the Fourier transform of $h(q)$. If $-1 < -\alpha < 0$, then we can even write

$$\psi(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-izq}f(q)\, dq, \quad z = p - i\alpha \tag{3.28}$$

and the integral is in fact absolutely convergent. Furthermore, for $-1 < \operatorname{Im} z < 0$, (3.28) defines an *analytic function* of z. Finally, the Fourier transforms of $f(q)$ and $h(q)$, namely $\psi(p)$ and $\psi(p - i)$, are boundary values, in the l.i.m. sense, of the analytic function $\psi(z)$ as $\operatorname{Im} z$ approaches 0 and -1, respectively. It should also be mentioned that, for $-1 < \operatorname{Im} z < 0$, $\psi(z)$ is square integrable with respect to $\operatorname{Re} z = p$ from $-\infty$ to $+\infty$.

[9] V. Bargmann, Ref. 3, p. 640.

[10] D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, New Jersey, 1941), Chap. VI, Sec. 8.

We conclude that if a vector f in H is such that both f and $K_+^{(0)}f$ have finite norms, then, in the p representation (J_2 diagonal) the wavefunction $f'(p)$ is the boundary value on the real axis (in the l.i.m. sense) of an analytic function $\psi(z)$, which is analytic at least for $-1 < \operatorname{Im} z < 0$; we can then define a new normalizable wavefunction, $f'(p-i)$, as the boundary value (in the l.i.m. sense) of $\psi(z)$ as $\operatorname{Im} z \to -1$; and we have

$$[K_+^{(0)}f]'(p) = f'(p-i). \tag{3.29}$$

The fact that both $f'(p)$ and $f'(p-i)$ are boundary values in the l.i.m. sense of the analytic function $\psi(z)$ is consistent with the fact that wavefunctions can be specified only up to sets of measure zero. However, this is all that is needed to completely specify the abstract vector in the Hilbert space. Written in terms of the ideal eigenvectors of J_2, we have

$$\begin{aligned} K_+^{(0)}\,|f\rangle &= K_+^{(0)}\int_{-\infty}^{\infty} f'(p)\,|p\rangle\,dp, \\ &= \int_{-\infty}^{\infty} f'(p-i)\,|p\rangle\,dp. \end{aligned} \tag{3.30}$$

If $|f\rangle$ is also in the domain of J_2, then we can write

$$J_2\,|f\rangle = J_2\int_{-\infty}^{\infty} f'(p)\,|p\rangle\,dp = \int_{-\infty}^{\infty} pf'(p)\,|p\rangle\,dp. \tag{3.31}$$

Thus we see that when J_2 is diagonal, it is meaningless to talk of the effect of applying the raising operator $K_+^{(0)}$ to an ideal eigenvector of J_2. $K_+^{(0)}$ may be applied only to those "linear combinations" of the ideal eigenvectors of J_2, which are vectors of finite norm, and are such that the wavefunctions $f'(p)$ appearing in the linear combination permit a unique analytic continuation into the lower half of the complex plane up to at least unit distance away from the real axis. Then the effect of $K_+^{(0)}$ is given by Eqs. (3.29) or (3.30). If a vector $|f\rangle$ is such that both J_2 and $K_+^{(0)}$ and their products may be applied to it, then we can use Eqs. (3.30) and (3.31) and check explicitly that

$$[J_2, K_+^{(0)}]\,|f\rangle = iK_+^{(0)}\,|f\rangle. \tag{3.32}$$

The situation is similar for the case of $K_-^{(0)}$, except that analytic continuations of wavefunctions into the upper half plane are involved. As for the operators $J_0^{(0)}$ and $J_1^{(0)}$, they may only be applied to states with wavefunctions permitting analytic continuations into both the upper and lower halves of the complex plane. (The eigenvalues of J_2 lie along the real axis in this complex plane!)

Returning to the more complicated but realistic case of the operators $K_\pm$, or J_0 and J_1, we do not examine them in any detail, but limit ourselves to a few comments. In this case, two new features appear, namely we have the multiplicity label r, as well as the differential operators d/dq in J_0 and J_1. However, the basic feature that the domains of J_0 and J_1 consist of states with boundary values of analytic functions as wavefunctions, in the basis where J_2 is diagonal, remains, and it is this that permits a simple understanding of the commutation rules (II.5).[11] Operating formally with (3.20), we find

$$\begin{aligned} [K_+f]_r'(p) &= \pm\left(p - s - \frac{i}{2}\right)f_r'(p-i), \\ [K_-f]_r'(p) &= \pm\left(p + s + \frac{i}{2}\right)f_r'(p+i). \end{aligned} \tag{3.33}$$

(The plus sign corresponds to $r = 1$, the minus sign to $r = 2$.) These equations may be thought of as the equivalents, when J_2 is diagonal, to the equations that are valid when J_0 is diagonal:

$$J_\pm\,|m\rangle = (\pm im + s + \tfrac{1}{2}i)\,|m \pm 1\rangle. \tag{3.34}$$

These are the same as (2.2) except for some phase changes in the basis states. Note also that (3.33) is a statement using wavefunctions, while (3.34) involves basis states. More details on these and related points will be discussed elsewhere.

IV. REPRESENTATIONS OF THE DISCRETE CLASS

In this section, we consider the UIR's of $SU(1,1)$ of the type D_k^+, $k = 1, 2, \cdots$. These may be realized via unitary transformations in a Hilbert space H_k of analytic functions of a complex variable z.[12] Elements of H_k correspond to functions $f(z)$ which are analytic and free of singularities in the open unit circle $|z| < 1$. The scalar product and norm are defined as follows:

$$\begin{aligned} (f, h)_k &= \frac{2k-1}{\pi}\int (1 - |z|^2)^{2k-2}\,\overline{f(z)}h(z)\,d^2z, \\ \|f\|_k &= (f,f)_k^{\frac{1}{2}} < \infty. \end{aligned} \tag{4.1}$$

Since the definition of the inner product depends on k, we have added the subscript k to the expressions above. The integration extends over the interior of the unit circle. The unitary transformation $U(g)$ representing an element g of $SU(1,1)$ is given by

$$[U(g)f](z) = [\bar\alpha + i\beta z]^{-2k} f\left(\frac{\alpha z - i\bar\beta}{\bar\alpha + i\beta z}\right). \tag{4.2}$$

[The connection between g and α, β is given in (1.1).] The expressions for the generators can be found by

[11] Cf. the remarks by E. C. G. Sudarshan, Ref. 1.

[12] V. Bargmann, Ref. 3, p. 623.

taking g to be the special elements corresponding to the matrices (1.9). They are

$$[J_0f](z) = kf(z) + z(df(z)/dz),$$
$$[J_1f](z) = -ikzf(z) + \tfrac{1}{2}i(1 - z^2)(df(z)/dz), \quad (4.3)$$
$$[J_2f](z) = -kzf(z) - \tfrac{1}{2}(1 + z^2)(df(z)/dz).$$

The orthonormal basis states $|m\rangle$ of (2.1) correspond to positive integral powers of z:

$$|m\rangle \rightarrow \left[\frac{(k+m-1)!}{(m-k)!\,(2k-1)!}\right]^{\frac{1}{2}} z^{m-k},$$
$$m = k, k+1, \cdots. \quad (4.4)$$

It is convenient to effect the diagonalization of J_2 in two stages. In the first stage, we change the complex variable z to another complex variable ω, by the formula

$$z = i\frac{\omega - i}{\omega + i}, \quad \omega = -i\frac{z+i}{z-i}. \quad (4.5)$$

The purpose of this transformation is to map the interior of the unit circle in the z plane onto the open upper halfplane in ω:

$$|z| < 1 \Rightarrow \operatorname{Im}\omega > 0. \quad (4.6)$$

The circumference of the unit circle in z is mapped onto the real line in ω: if we write $z = e^{i\varphi}$, then $-\frac{1}{2}\pi \le \varphi \le \frac{1}{2}\pi$ corresponds to $0 \le \omega < \infty$, and $\frac{1}{2}\pi \le \varphi \le \frac{3}{2}\pi$ corresponds to $-\infty < \omega \le 0$. The imaginary z axis, $z = i\rho$, $-1 \le \rho \le 1$, goes into the whole of the positive imaginary axis in ω.

The change in wavefunction from an analytic function $f(z)$ to an analytic function in ω is defined by[13]

$$f(z) \rightarrow f_1(\omega) = \frac{1}{(\omega + i)^{2k}} f(z). \quad (4.7)$$

[Both wavefunctions $f(z)$ and $f_1(\omega)$ represent the same abstract vector f in H_k.] Using (4.7), we express the scalar product and the generators in terms of ω[14]:

$$(f, h)_k = \frac{2k-1}{\pi} 4^{2k-1} \int_{-\infty}^{\infty} d(\operatorname{Re}\omega)$$
$$\times \int_0^{\infty} d(\operatorname{Im}\omega)\overline{f_1(\omega)}h_1(\omega)(\operatorname{Im}\omega)^{2k-2}, \quad (4.8)$$

$$[J_0f]_1(\omega) = -ik\omega f_1(\omega) - \tfrac{1}{2}i(1+\omega^2)\frac{df_1(\omega)}{d\omega},$$
$$[J_1f]_1(\omega) = -ik\omega f_1(\omega) + \tfrac{1}{2}i(1-\omega^2)\frac{df_1(\omega)}{d\omega}, \quad (4.9)$$
$$[J_2f]_1(\omega) = ikf_1(\omega) + i\omega\frac{df_1(\omega)}{d\omega}.$$

[13] We have added a subscript to $f_1(\omega)$ to distinguish this function from the original function $f(z)$.

[14] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic Press Inc., New York, 1966), Vol. 5, Chap. VII, give the discrete representations in this form. The negative integer s used in this reference to label the representations is related to k by: $s = 1 - 2k$.

It is clear that the wavefunctions $f_1(\omega)$ corresponding to vectors in H_k are analytic and free from singularities in the upper ω half plane. As a matter of fact, in addition to being analytic for $\operatorname{Im}\omega > 0$, the admissible functions $f_1(\omega)$ must obey certain extra conditions, related to the behavior of $f_1(\omega)$ on the real axis and at the point at infinity.[15]

In order to effect the diagonalization of J_2, we need to use a special representation of the functions $f_1(\omega)$. For this, we momentarily restrict ourselves to a dense subset of H_k which would correspond, in the description in terms of the variable z, to functions which can be analytically continued outside the unit circle [in addition to being normalizable in the sense of (4.1)]. For example, such a dense subset of H_k is provided by the set of all finite power series in z, or the set of all entire functions of z. The normalizability of the corresponding functions $f_1(\omega)$, together with their analyticity properties, imply that the $f_1(\omega)$ can be represented for all ω in terms of their boundary values as ω approaches the positive real axis from above. This is achieved via the Mellin transform of $f_1(\omega)$.[16] By using this representation, we can rewrite the expression for the scalar product and the norm of elements in H_k in a form which involves only the values of $f_1(\omega)$ on the positive real axis. Having obtained this form, we can then recover all of H_k from the dense subset of H_k by the standard method of completion. Explicitly, then, $f_1(\omega)$ may be represented in the following fashion:

$$f_1(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, \omega^{ip-k} \int_0^{\infty} f_1(x + i\epsilon)x^{k-ip-1}\, dx. \quad (4.10)$$

Here, we use the variable x when ω approaches the positive real axis from above, and the complex power of ω is defined unambiguously for $\operatorname{Im}\omega > 0$ by the convention:

$$\omega^{ip-k} = e^{(ip-k)\ln\omega},$$
$$\ln\omega = \ln|\omega| + i\arg\omega, \quad 0 < \arg\omega < \pi. \quad (4.11)$$

Actually, the variable x is not the most natural one for our purposes. We instead use q defined by

$$x = e^q; \quad -\infty < q < \infty, \quad (4.12)$$

so that the positive real x axis corresponds to the entire real axis in q. At the same time, we use in place of $f_1(x + i\epsilon)$ a new wavefunction in terms of q, given by

$$\tilde{f}(q) = x^k f_1(x + i\epsilon). \quad (4.13)$$

The result of all these manipulations is to yield an

[15] I. M. Gel'fand *et al.*, Ref. 14, Chap. VII, p. 423.

[16] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford University Press, London, 1948), p. 46.

expression for the analytic functions $f_1(\omega)$ in terms of the new wavefunctions $\tilde{f}(q)$:

$$f_1(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, \omega^{ip-k}\int_{-\infty}^{\infty} dq\, e^{-ipq}\tilde{f}(q). \quad (4.14)$$

We must now use (4.14) to write the scalar product of two vectors in H_k in terms of their wavefunctions in the q basis. It is, however, instructive to first evaluate the forms of the generators J_0, J_1, and J_2, as they act on wavefunctions $\tilde{f}(q)$, and then rewrite the scalar product. For this purpose, we need to restrict ourselves to wavefunctions $\tilde{f}(q)$ lying in the domain of one of these generators such that its Fourier transform permits certain analytic continuations into the complex plane. Such wavefunctions of course constitute a dense subset of H_k. We find quite easily:

$$\begin{aligned}
[J_0 f]^{\sim}(q) &= (1/i)\cosh q[d\tilde{f}(q)/dq] - ik\sinh q\tilde{f}(q),\\
[J_1 f]^{\sim}(q) &= i\sinh q[d\tilde{f}(q)/dq] + ik\cosh q\tilde{f}(q), \quad (4.15)\\
[J_2 f]^{\sim}(q) &= (1/i)\, d\tilde{f}(q)/dq.
\end{aligned}$$

Before going any further, let us remark that these forms for the generators are practically identical to those in (3.11), except that the parameter s in the latter equations has been "analytically continued" to a complex value $-ik + \frac{1}{2}i$. Thus the representation of vectors by wavefunctions in the q variable, as introduced above, is the analogue for the representations D_k^+ to the q representation we found for the representations C_q^0 in Sec. III.

The generators (3.11) for the C_q^0 representations are symmetric (i.e., Hermitian) operators with respect to the ordinary local scalar product in q space, Eq. (3.9). However, this is so only as long as s is real, and we should expect that the metric in q space, with respect to which the generators (4.15) are formally Hermitian, will be a nonlocal (though positive definite and translation invariant) one. We now proceed to work out the expression for the scalar product. We use the representation (4.14) in (4.8), and use polar coordinates for the ω integration in (4.8). With the aid of the formula[17]

$$\int_0^{\pi} d\theta\, e^{-2p\theta}(\sin\theta)^{2k-2} = \frac{\pi}{4^{k-1}}\frac{e^{-\pi p}\Gamma(2k-1)}{\Gamma(k+ip)\Gamma(k-ip)}, \quad (4.16)$$

we derive

$$(f, h)_k = 4^{k-1}\frac{\Gamma(2k)}{\pi}\int_{-\infty}^{\infty} dp[\Gamma(k+ip)\Gamma(k-ip)]^{-1}e^{-\pi p} \times\left\{\int_{-\infty}^{\infty} dq'\, e^{ipq'}\overline{\tilde{f}(q')}\right\}\left\{\int_{-\infty}^{\infty} dq\, e^{-ipq}\tilde{h}(q)\right\}. \quad (4.17)$$

It is tempting to try and write this in the form

$$(f, h)_k = \int_{-\infty}^{\infty} dq'\int_{-\infty}^{\infty} dq\,\overline{\tilde{f}(q')}K(q'-q)\tilde{h}(q), \quad (4.18)$$

with the kernel $K(q)$ being the expression

$$K(q) = 4^{k-1}\frac{\Gamma(2k)}{\pi}\int_{-\infty}^{\infty} e^{-\pi p}e^{ipq}\frac{dp}{\Gamma(k+ip)\Gamma(k-ip)}. \quad (4.19)$$

If this were possible, then the kernel $K(q)$ would be the representative in q space of a *local*, *positive definite* metric in p space with a real, positive definite weight function $\rho(p)$, where

$$\rho(p) = 4^{k-1}\frac{\Gamma(2k)}{\pi}(e^{-\pi p}/\Gamma(k+ip)\Gamma(k-ip)). \quad (4.20)$$

However, this way of writing the metric in q space, namely as in (4.18), must be viewed as just a formal rewriting of (4.17), since $\rho(p)$ increases too fast as $p \to -\infty$ to allow $K(q)$ to exist as an ordinary function. Strictly speaking, then one just finds the forms (4.15) for the generators, together with the metric (4.17), in the q basis.

In spite of this situation in the q basis, the diagonalization of J_2 can be effected, and as a matter of fact the scalar product (4.17) already appears in a form that can be directly interpreted as a positive–definite local scalar product in the p space (J_2 diagonal). We define the p space wavefunctions by

$$f'(p) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-ipq}\tilde{f}(q)\, dq, \quad (4.21)$$

and the scalar product is

$$(f, h)_k = 2\pi\int_{-\infty}^{\infty}\rho(p)\overline{f'(p)}h'(p)\, dp. \quad (4.22)$$

J_2 is diagonal in this basis:

$$[J_2 f]'(p) = pf'(p). \quad (4.23)$$

It seems that the existence of a complicated expression for the scalar product is forced on us if we wish to have the generators in as close a form as possible to the case of the UIR's C_q^0, namely in the form (4.15). It should be possible to introduce a basis (q' basis, say), with J_2 being the generator of translations in q', in such a way that the scalar product of two wavefunctions is given by the usual local expression, i.e., with no nonlocal kernel appearing in the q' integration. However, in such a basis, the generators J_0 and J_1 would be considerably different from (4.15). We hope to discuss these questions, as also the precise properties of the permissible wavefunctions $\tilde{f}(q)$, and

[17] *Bateman Manuscript Project* (California Institute of Technology, Pasadena, California, 1953), Vol. 1, p. 12.

the nature of the Fourier-transform operation (4.21) in the context of the scalar products (4.17) and (4.22), on another occasion.

We conclude by noting that according to the considerations above, the spectrum of J_2 consists of the entire real line, and that every eigenvalue of J_2 occurs just once. This is in accord with the statement of Bargmann[18] and is characteristic of the representations of the discrete classes, $D_k^{\pm}$. Finally, the effects of the generators J_0 and J_1 on a wavefunction $f'(p)$ which can be analytically continued in p is given by equations exactly like (3.33) except that s is replaced by $-ik + \frac{1}{2}i$.

CONCLUSION

In this paper, we have attempted to cast some of the unitary irreducible representations of the group $SU(1, 1)$ into a form in which the generator of a noncompact subgroup $O(1, 1)$ is diagonal. For the representations of the continuous class, we have recovered the result that the spectrum of this noncompact generator consists of the entire real line, and that each eigenvalue appears twice; for the representations of the discrete class we have found the expected result that the spectrum of the noncompact generator is again the real line, but with no multiplicity of eigenvalues. We have also seen that when the noncompact generator is diagonal, the domain of the other generators of $SU(1, 1)$ consists of vectors whose wavefunctions are boundary values of analytic functions, and thus permit a unique continuation into the complex plane. In this way we have been able to understand the rather unusual commutation relations which at first sight suggest that the eigenvalues of a self-adjoint operator could be shifted by an imaginary amount $\pm i$.

It is interesting to note that in a suitable basis related in a simple way to the diagonalization of a noncompact generator, the generators of $SU(1, 1)$ in both the continuous-class representations and the discrete-class representations, (3.11) and (4.15), have the same analytic form, the only difference being in the parameter related to the Casimir invariant of the representation. In a sense, this is analogous to the fact that in the $O(2)$ basis, the matrix elements of the generators can be given in a universal fashion, valid for all classes of representations [Eq. (2.2)]. This is in the spirit of the method of the Master Analytic Representation.[19]

ACKNOWLEDGMENTS

The author has greatly benefited from many enlightening discussions with Professor J. G. Kuriyan, Professor L. O'Raifeartaigh, and Professor E. C. G. Sudarshan.

Most of this work was done while the author was at the Palmer Physical Laboratory, Princeton University, during the academic year 1965–1966. He would like to thank Professor W. Bleakney and Professor M. L. Goldberger for the hospitality of that Institution.

[18] V. Bargmann, Ref. 3, p. 640.

[19] J. G. Kuriyan, Ref. 7; J. G. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, Ref. 7.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Geometric Theory of the Spin-Weighted Functions*

A. STARUSZKIEWICZ†
Syracuse University, Syracuse, New York

(Received 21 March 1967)

The spin-weighted functions introduced recently are shown to be eigenfunctions of the total angular momentum for appropriately defined geometric objects on the sphere, namely the Pensov objects.

1. INTRODUCTION

In a recent work[1] it has been found useful to introduce a class of functions ${}_sY_{lm}$ defined on the sphere. The ${}_sY_{lm}$ can be considered as generalized spherical functions. These functions appear also in the theory of representations of the rotation group.[2,3] Our aim is to formulate a geometric theory of the ${}_sY_{lm}$. Our method is the following: The group of rotations is characterized by three operators generating infinitesimal motions and by their Lie algebra. Realization of the operators is not unique. A general method of constructing various realizations consists in the following: We choose an arbitrary geometric quantity on the sphere and calculate the Lie derivative with respect to generators of infinitesimal motions (i.e., Killing vectors). Commutation relations of the Lie derivatives do not depend on a geometric quantity, but only on the generating vectors.[4,5] Accordingly, every geometric quantity ϕ (indices suppressed) on the sphere generates a particular realization of the Lie algebra of the group of rotations. We can associate with every realization a set of orthogonal functions, namely the solutions of the eigenvalue problem

$$-\left(\underset{\xi_1}{\pounds}^2 + \underset{\xi_2}{\pounds}^2 + \underset{\xi_3}{\pounds}^2\right)\phi = \lambda\phi,$$
$$-i\,\underset{\xi_3}{\pounds}\,\phi = \sigma\phi, \tag{1.1}$$

where $\pounds_{\xi_n}$ denotes the Lie derivative with respect to the nth Killing vector. If for example ϕ is an invariant, Eq. (1.1) determines the usual spherical functions. Accordingly, our problem is to identify that particular geometric quantity which generates the ${}_sY_{lm}$. It follows from the nature of the ${}_sY_{lm}$ that it should be a one-component geometric quantity.

2. THE PENSOV OBJECTS

Classification of one-component differential geometric objects has been given by Pensov.[6] He finds that in V_2 (and only in V_2) there exists nonlinear objects with the transformation rule[5]

$$\overset{(n')}{\Omega} = \frac{A_1^{1'}\overset{(n)}{\Omega} + A_2^{1'}}{A_1^{2'}\overset{(n)}{\Omega} + A_2^{2'}},$$

where $A_\lambda^{n'} = \partial_\lambda x_\lambda^{n'}$. The ratio of the components of a contravariant vector is an example of the object Ω. We shall call objects similar to Ω the Pensov objects. Any function of the form

$$\Omega = \frac{av^1 + bv^2}{cv^1 + dv^2}$$

where v^μ are contravariant components of a vector and $ad - bc \neq 0$ is also a geometric object. The case $a = c = 1$, $d = -b = i = -1^{\frac{1}{2}}$ is particularly interesting. The transformation rule of the object in this case has the form

$$\overset{(n')}{\Omega} = \frac{[A_1^{1'} + A_2^{2'} - i(A_1^{2'} - A_2^{1'})]\overset{(n)}{\Omega} + A_1^{1'} - A_2^{2'} - i(A_1^{2'} + A_2^{1'})}{[A_1^{1'} - A_2^{2'} + i(A_1^{2'} + A_2^{1'})]\overset{(n)}{\Omega} + A_1^{1'} + A_2^{2'} - i(A_2^{1'} - A_1^{2'})}. \tag{2.1}$$

It is a well known fact that every two-dimensional Riemannian space is conformally flat. Consequently, we can without loss of generality assume the line element of V_2 to be of the form

$$ds^2 = f(x^1, x^2)[(dx^1)^2 + (dx^2)^2] \tag{2.2}$$

and restrict our transformation group to the subgroup of conformal motions of the line element (2.2). Conformal motions satisfy the Cauchy–Riemann conditions

$$A_1^{1'} = A_2^{2'}, \quad A_2^{1'} = -A_1^{2'}.$$

In this case the transformation rule (2.1) assumes the form

$$\overset{(n')}{\Omega} = \frac{A_1^{1'} - iA_1^{2'}}{A_1^{1'} + iA_1^{2'}}\overset{(n)}{\Omega}.$$

* Supported by Air Force Office of Scientific Research.
† On leave from the Institute of Physics, Jagellonian University, Cracow.
[1] E. T. Newman and R. Penrose, J. Math. Phys. 7, 863 (1966).
[2] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan, J. Math. Phys. 8, 2155 (1967).
[3] I. M. Gel'fand, R. A. Minlos, and Z. Y. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (The Macmillan Company, New York, 1963), p. 83.
[4] J. A. Schouten, *Ricci Calculus* (Springer-Verlag, Berlin, 1954), p. 110.
[5] In accordance with the notation of J. A. Schouten, Ref. 4, p. 12, the superscripts (n) and (n') refer the object Ω to the coordinates x^n and $x^{n'}$, respectively.
[6] G. Pensov, Compt. Rend. Acad. Sci. URSS **54**, 563 (1946).

This means that with respect to the subgroup of conformal motions the object Ω is linear and homogeneous, i.e., is a geometric quantity.[7] It is clear that we can introduce a set of generalized quantities $\Omega_{(s)} = \Omega^{s/2}$, where s is an arbitrary number, with the transformation rule

$$\overset{(n')}{\Omega_{(s)}} = \left(\frac{A_1^{1'} - iA_1^{2'}}{A_1^{1'} + iA_1^{2'}}\right)^{s/2} \overset{(n)}{\Omega_{(s)}}.$$

3. THE COVARIANT AND LIE DERIVATIVES OF PENSOV OBJECTS

It is convenient to introduce a complex coordinate system

$$y^1 = x^1 + ix^2, \quad y^{1'} = x^{1'} + ix^{2'},$$
$$y^2 = x^1 - ix^2, \quad y^{2'} = x^{1'} - ix^{2'}.$$

In a new coordinate system the Cauchy–Riemann conditions assume the form $A_2^{1'} = A_1^{2'} = 0$, where now $A_\lambda^{n'} = \partial_\lambda y^{n'}$. The transformation rule of the object $\Omega_{(s)}$ has the form

$$\overset{(n')}{\Omega} = \left(\frac{A_2^{2'}}{A_1^{1'}}\right)^{s/2} \overset{(n)}{\Omega_{(s)}}. \tag{3.1}$$

Let us differentiate Eq. (3.1) with respect to $y^{\mu'}$. Taking into account the Cauchy–Riemann conditions and the well-known formula

$$\frac{\partial A_{\delta'}^\lambda}{\partial x^{\mu'}} = \Gamma_{\mu'\delta'}^{\lambda'} A_{\lambda'}^\lambda - \Gamma_{\mu\nu}^\lambda A_{\mu'}^\mu A_{\delta'}^\nu,$$

where Γ is the object of parallel displacement, we obtain

$$\partial_{\mu'} \overset{(n')}{\Omega_{(s)}} = \frac{s}{2}(\Gamma_{1'\mu'}^{1'} - \Gamma_{2'\mu'}^{2'}) \overset{(n')}{\Omega_{(s)}} - A_{\mu'}^\lambda \left(\frac{A_2^{2'}}{A_1^{1'}}\right)^{s/2} \left[\partial_\lambda \overset{(n)}{\Omega_{(s)}} - \frac{s}{2}(\Gamma_{1\lambda}^1 - \Gamma_{2\lambda}^2) \overset{(n)}{\Omega_{(s)}}\right].$$

Hence, we can define the covariant derivative of the object $\Omega_{(s)}$ as

$$\nabla_\mu \Omega_{(s)} = \partial_\mu \Omega_{(s)} - \frac{s}{2}(\Gamma_{1\mu}^1 - \Gamma_{2\mu}^2)\Omega_{(s)}.$$

The Lie derivative of the object $\Omega_{(s)}$ is easily found to be

$$\underset{\xi}{£} \Omega_{(s)} = \xi^\lambda \partial_\lambda \Omega_{(s)} + \frac{s}{2}(\partial_1 \xi^1 - \partial_2 \xi^2)\Omega_{(s)}.$$

The Lie derivative of a Pensov object is a Pensov object with the same index s, as it of course, should be.

4. OPERATORS OF ANGULAR MOMENTUM

Having established the form of the Lie derivative of Pensov objects we can define the operators of angular momentum

$$\underset{n}{J}\, \Omega_{(s)} = -i \underset{\xi_n}{£} \Omega_{(s)},$$

where ξ_n^μ, $n = 1, 2, 3$ are the generators of infinitesimal motions of the sphere (i.e., Killing vectors). Equation (1.1) is in this case an eigenequation for the total angular momentum. It turns out that solutions of this equation are the ${}_sY_{lm}$ up to a constant factor.

To prove this it is convenient to introduce the coordinate system used in Newman and Penrose's paper,[1] namely,

$$y^1 = \zeta = e^{i\phi} \cot \tfrac{1}{2}\theta, \quad y^2 = \bar{\zeta} = e^{-i\phi} \cot \tfrac{1}{2}\theta,$$

where θ and ϕ are the usual spherical coordinates. In this coordinate system,

$$\left(\underset{\xi_1}{£} + i \underset{\xi_2}{£}\right)\Omega_{(s)} = -i\left(\zeta^2 \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}} + s\zeta\right)\Omega_{(s)},$$

$$\left(\underset{\xi_1}{£} - i \underset{\xi_2}{£}\right)\Omega_{(s)} = i\left(\frac{\partial}{\partial \zeta} + \bar{\zeta}^2 \frac{\partial}{\partial \bar{\zeta}} - s\bar{\zeta}\right)\Omega_{(s)},$$

$$\underset{\xi_3}{£} \Omega_{(s)} = i\left(\zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} + s\right)\Omega_{(s)}.$$

Using these expressions, one can easily calculate the left-hand side of the first equation (1.1)

$$-\left(\underset{\xi_1}{£^2} + \underset{\xi_2}{£^2} + \underset{\xi_3}{£^2}\right)\Omega_{(s)} = \left[-(2P)^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} + 2sP\left(\zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}}\right) + 2s^2 P\right]\Omega_{(s)} = \lambda\, \Omega_{(s)},$$

where $2P = 1 + \zeta\bar{\zeta}$. The solution of this equation [and of the second equation (1.1)], regular for $\zeta \to \infty$, can be written down by means of the hypergeometric function

$$\Omega_{(s)} = \left(\frac{\zeta}{\bar{\zeta}}\right)^{m/2} \left(\frac{1}{2P}\right)^{|s+m/2|} \left(1 - \frac{1}{2P}\right)^{|m/2|} F\left(a, b; |2s + m| + 1; \frac{1}{2P}\right), \tag{4.1}$$

where m is an integer, $a + b = 2\gamma + 1$, $ab = \gamma(\gamma + 1) - \lambda$, and $\gamma = |s + \frac{1}{2}m| + |\frac{1}{2}m|$. It should be noted that s as a geometric characteristic of a Pensov object can be an arbitrary number. However, the solution (4.1) is finite everywhere if and only if $\Omega_{(s)}$ is a polynomial. It is easy to see that this can be the case for positive and negative m if and only if s is an integer or an integer plus $\frac{1}{2}$. Accordingly, Pensov objects as geometric quantities can have an arbitrary index s, but they can be interpreted as quantum-mechanical wavefunctions only for s equal to an integer or an integer plus $\frac{1}{2}$.

It is easy to see that polynomial solutions (4.1) are the spin-weighted functions up to a constant factor.

[7] J. A. Schouten, Ref. 4, p. 68.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Finite-Range Effects in Distorted-Wave Born-Approximation Calculations of Nucleon Transfer Reactions*

T. SAWAGURI AND W. TOBOCMAN
Physics Department, Case Western Reserve University, Cleveland, Ohio

(Received 21 February 1967; final manuscript received 10 April 1967)

A method for doing distorted-wave Born-approximation calculations for nucleon transfer reactions is presented. This method is designed to be used when the zero-range approximation cannot be made. The method has the advantage that only simple quadratures need be performed. When recoil effects are negligible, our method leads to a particularly simple form. The technique developed here can be applied to the evaluation of the six-dimensional integrals that result when the general two-body interaction is considered. Thus it may be usefully applied to nuclear-structure calculations when it is desired to use wavefunctions which are not harmonic-oscillator eigenfunctions.

I. INTRODUCTION

Distorted-wave Born-approximation calculations of the nucleon transfer reaction amplitude are only easily carried out when it can be assumed that the interaction potential binding the transferred nucleon in the initial or final state is of zero range. This approximation is probably adequate for deutron-stripping reactions. However, for reactions in which a nucleon is transferred between two heavy nuclei, the zero-range approximation cannot be justified.

Consider a nuclear reaction in which a nucleon N is transferred from nucleus $I = A + N$ to nucleus $F = B + N$. Then the distorted-wave Born-approximation (DWBA) amplitude is

$$A_{IF} = \int d^3r_{NA} \int d^3r_{IB} \Phi_{AF}^{(-)}(\mathbf{K}_A, \mathbf{r}_{AF})^* \phi_{NB}(\mathbf{r}_{NB})^* \times V_{AN}(\mathbf{r}_{AN}) \phi_{AN}(\mathbf{r}_{AN}) \Phi_{IB}^{(+)}(\mathbf{K}_I, \mathbf{r}_{IB}), \quad (1)$$

where ϕ_{AN} is the initial nucleon bound state, $\Phi_{IB}^{(+)}$ is the relative-motion wavefunction in the incident channel, and ϕ_{NB} and $\Phi_{AF}^{(-)}$ are the corresponding quantities for the final bound state and outgoing channel. By setting $V_{AN}\phi_{AN}$ equal to a delta function, the transition amplitude becomes a three-dimensional integral. This can be reduced to a sum of one-dimensional radial integrals by expanding all the wavefunctions in spherical harmonics and carrying out the angular integrations. Without this zero-range approximation, we are left with a difficult multiple integral to perform. A procedure for carrying out this multiple integral has been presented by Austern *et al.*[1]

We wish to suggest an alternative approach to the evaluation of the finite-range DWBA amplitude. In our method $V_{AN}\phi_{AN}$ is expanded in an infinite sum of products of functions of $\mathbf{r}_{IB}$ and functions of $\mathbf{r}_{AF}$ or $\mathbf{r}_{NB}$. This leads to an expression for the amplitude A_{IF} which is an infinite sum of products of one-dimensional radial integrals. This sum converges rapidly enough, we believe, to make this a practical method for calculating the transition amplitude.

There have been other finite-range treatments of the nucleon-transfer amplitude, but these have not been based on the unmodified DWBA. Breit[2] has given a semiclassical treatment. Dar and Kozlowsky[3] and Dodd and Greider[4] have given treatments based on the diffraction model. Buttle and Goldfarb[5] have devised a DWBA treatment for the process in which the bound-state functions are approximated by their asymptotic forms.

The technique developed here can be applied to the evaluation of the six-dimensional integrals that result when the general two-body interaction is considered. Thus it may be usefully applied to nuclear-structure calculations when it is desired to use wavefunctions which are not harmonic-oscillator eigenfunctions.

In Sec. II we show how a function of $\mathbf{r}_1 - \mathbf{r}_2$ can be expanded in an infinite series of products of functions of $\mathbf{r}_1$ and functions of $\mathbf{r}_2$. The functions used in the expansion are modifications of the harmonic-oscillator eigenfunctions. The expansion coefficients have a relatively simple form. The DWBA expression for the nucleon-transfer amplitude in the no recoil limit is given in Sec. III. The interaction is assumed to have finite range. It is shown that the evaluation of the amplitude requires only the performance of simple quadratures. Section IV is devoted to a discussion of how recoil effects might be included. In Sec. V we examine the zero-range limit of our expansion for the transition amplitude. We find that it

*This work supported by the Atomic Energy Commission.

[1] N. Austern, R. M. Drisko, E. C. Halbert, and G. R. Satchler, Phys. Rev. **133**, B3 (1964).

[2] G. Breit and M. E. Ebel, Phys. Rev. **103**, 679 (1956); G. Breit, *ibid.* **135**, B1323 (1964).

[3] A. Dar and B. Kozlowsky, Phys. Rev. Letters **15**, 1036 (1965).

[4] L. R. Dodd and K. R. Greider, Phys. Rev. Letters **14**, 959 (1965).

[5] P. J. A. Buttle and L. J. B. Goldfarb, Nucl. Phys. **78**, 409 (1966).

does not seem to be simply related to the unexpanded zero-range amplitude expression.

II. THE EXPANSION METHOD

Our basic problem is to find a way of expanding an arbitrary function of $\mathbf{r}_1 - \mathbf{r}_2$ in a sum of products of functions of $\mathbf{r}_1$ and $\mathbf{r}_2$. The only function F possessing the property $F(\mathbf{r}_1 - \mathbf{r}_2) = F(\mathbf{r}_1)F(-\mathbf{r}_2)$ is the exponential. Therefore we start by Fourier analyzing the given function $\Phi(\mathbf{r}_1 - \mathbf{r}_2)$:

$$\Phi(\mathbf{r}_1 - \mathbf{r}_2) = \int d^3k e^{i\mathbf{k}\cdot\mathbf{r}_1} e^{-i\mathbf{k}\cdot\mathbf{r}_2} \int d^3r \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^3} \Phi(\mathbf{r}). \quad (2)$$

Next let us expand the plane waves in the integrand in terms of spherical harmonics and perform the angular integrations:

$$\Phi(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{\substack{\ell\ell_1\ell_2 \\ mm_1m_2}} Y_{\ell_1}^{m_1}(\Omega_1) Y_{\ell_2}^{m_2}(\Omega_2)^* \begin{Bmatrix} \ell_1 m_1 \\ \ell_2 m_2 \\ \ell m \end{Bmatrix} \int_0^\infty dk k^2$$
$$\times j_{\ell_1}(kr_1) j_{\ell_2}(kr_2) \int_0^\infty dr r^2 j_\ell(kr) \int d\Omega Y_\ell^m(\Omega)^* \Phi(\mathbf{r}), \quad (3a)$$

where Y_ℓ^m is the normalized spherical harmonic function, j_ℓ is the spherical Bessel function, and

$$\begin{Bmatrix} \ell_1 m_1 \\ \ell_2 m_2 \\ \ell m \end{Bmatrix} = i^{(\ell_1 - \ell_2 - \ell)}\, 8\left[\frac{(2\ell_2+1)(2\ell+1)}{4\pi(2\ell_1+1)}\right]^{\frac{1}{2}}$$
$$\times (\ell_2 \ell m_2 m \mid \ell_2 \ell \ell_1 m_1)(\ell_2 \ell 00 \mid \ell_2 \ell \ell_1 0). \quad (3b)$$

$(\ell_1\ell_2 m_1 m_2 \mid \ell_1 \ell_2 \ell m)$ is the vector addition coefficient.

Now we expand the spherical Bessel functions j_{ℓ_1} and j_{ℓ_2} in terms of a discrete set of functions $\mathcal{F}_{n_i}^{\ell_i}(\alpha_i, \beta r_i)$:

$$j_{\ell_i}(kr_i) = \sum_{n_i} \mathcal{C}_{n_i}^{\ell_i}(\alpha_i, k/\beta_i) \mathcal{F}_{n_i}^{\ell_i}(\alpha_i, \beta_i r_i). \quad (4)$$

The result is

$$\Phi(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{\substack{\ell_1 m_1 n_1 \\ \ell_2 m_2 n_2 \\ \ell m}} Y_{\ell_1}^{m_1}(\Omega_1) Y_{\ell_2}^{m_2}(\Omega_2) \mathcal{F}_{n_1}^{\ell_1}(\alpha_1, \beta_1 r_1)$$
$$\times \mathcal{F}_{n_2}^{\ell_2}(\alpha_2, \beta_2 r_2) \begin{Bmatrix} \ell_1 m_1 \\ \ell_2 m_2 \\ \ell m \end{Bmatrix} \int_0^\infty dr r^2 \mathcal{G}_{n_1 n_2}^{\ell\ell_1\ell_2}(r) \int d\Omega\, Y^m(\Omega)^* \Phi(\mathbf{r}), \quad (5a)$$

where

$$\mathcal{G}_{n_1n_2}^{\ell\ell_1\ell_2}(r) = \int_0^\infty dk k^2 j_\ell(kr) \mathcal{C}_{n_1}^{\ell_1}\left(\alpha_1, \frac{k}{\beta_1}\right) \mathcal{C}_{n_2}^{\ell_2}\left(\alpha_2, \frac{k}{\beta_2}\right). \quad (5b)$$

We need to choose the complete set of functions $\mathcal{F}_n^\ell$ to be such that $\mathcal{G}_{n_1n_2}^{\ell\ell_1\ell_2}(r)$ has a convenient form.

A set of functions which seems to be well suited to our needs are what we will call the modified harmonic-oscillator (MHO) functions

$$\mathcal{F}_n^\ell(\alpha, \beta r) = \left[\frac{2\Gamma(n+\ell+\frac{3}{2})}{n!\Gamma(\ell+\frac{3}{2})^2}\right]^{\frac{1}{2}} (\beta r)^\ell$$
$$\times e^{-\frac{1}{2}\alpha\beta^2 r^2} {}_1F_1(-n; \ell+\tfrac{3}{2}; \beta^2 r^2), \quad (6)$$

where Γ is the Γ function and ${}_1F_1$ is the hypergeometric function. $\mathcal{F}_n^\ell(1, \beta r)$ is the radial harmonic-oscillator function.

The MHO functions enjoy the biorthogonal property proved in Appendix A:

$$\int_0^\infty d\rho\, \rho^2 \mathcal{F}_n^\ell(\alpha; \rho) \mathcal{F}_m^\ell(2-\alpha, \rho) = \delta_{n,m}. \quad (7)$$

This property is a simple consequence of the orthonormality of the radial harmonic-oscillator functions.

The spherical Bessel function expansion coefficient is evaluated in Appendix B. There it is shown that

$$\mathcal{C}_n^\ell(\alpha, k/\beta)$$
$$= \beta^3 \int dr\, r^2 j_\ell(kr) \mathcal{F}_n^\ell(2-\alpha, \beta r)$$
$$= (-1)^n (\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{\alpha^{n+\ell/2}}{(2-\alpha)^{n+\ell/2+\frac{3}{2}}} \mathcal{F}_n^\ell\left(\alpha; \frac{k}{\beta[\alpha(2-\alpha)]}\right). \quad (8)$$

Substitution of Eq. (8) into Eq. (5b) leads to an integral which is evaluated in Appendix C. The result is the following expression for $\mathcal{G}$:

$$\mathcal{G}_{n_1n_2}^{\ell\ell_1\ell_2}(r) = \sum_{s_1=0}^{n_1} \sum_{s_2=0}^{n_2} \begin{Bmatrix} s_1 n_1 \ell_1 \beta_1 \alpha_1 \\ s_2 n_2 \ell_2 \beta_2 \alpha_2 \\ n\ell\beta \end{Bmatrix} \mathcal{F}_n^\ell(2, \beta r), \quad (9a)$$

where

$$n = s_1 + s_2 + \tfrac{1}{2}(\ell_1 + \ell_2 - \ell), \quad (9b)$$

$$\beta = \left\{\frac{\beta_1^2(2-\alpha_1)\beta_2^2(2-\alpha_2)}{2[\beta_1^2(2-\alpha_1) + \beta_2^2(2-\alpha_2)]}\right\}^{\frac{1}{2}}, \quad (9c)$$

$$\begin{Bmatrix} s_1 n_1 \ell_1 \beta_1 \alpha_1 \\ s_2 n_2 \ell_2 \beta_2 \alpha_2 \\ n\ell\beta \end{Bmatrix}$$
$$= \left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{[n!\Gamma(n+\ell+\frac{3}{2})n_1! \times \Gamma(n_1+\ell_1+\frac{3}{2})n_2!\Gamma(n_2+\ell_2+\frac{3}{2})]^{\frac{1}{2}}}{2(n_1-s_1)!(n_2-s_2)!s_1!s_2! \times \Gamma(s_1+\ell_1+\frac{3}{2})\Gamma(s_2+\ell_2+\frac{3}{2})}$$
$$\times \frac{(-\alpha_1)^{n_1-s_1}(-\alpha_2)^{n_2-s_2}(2\beta)^{2n+\ell+3}}{(2-\alpha_1)^{n_1+s_1+\ell_1+\frac{3}{2}}(2-\alpha_2)^{n_2+s_2+\ell_2+\frac{3}{2}}\beta_1^{2s_1+\ell_1}\beta_2^{2s_2+\ell_2}}. \quad (9d)$$

Thus Eq. (5a) can be written

$$\Phi(\mathbf{r}_1-\mathbf{r}_2)=\sum_{\substack{\ell_1 m_1 n_1 \ell\\ \ell_2 m_2 n_2 m}} Y_{\ell_1}^{m_1}(\Omega_1)Y_{\ell_2}^{m_2}(\Omega_2)^*\mathcal{F}_{n_1}^{\ell_1}(\alpha_1,\beta_1 r_1)$$
$$\times\, \mathcal{F}_{n_2}^{\ell_1}(\alpha_2,\beta_2 r_2) \begin{Bmatrix}\ell_1 m_1\\ \ell_2 m_2\\ \ell m\end{Bmatrix}\sum_{s_1 s_2}\begin{Bmatrix} s_1 n_1\ell_1\beta_1\alpha_1\\ s_2 n_2 \ell_2\beta_2\alpha_2\\ n\ell\beta\end{Bmatrix}$$
$$\times\int_0^\infty dr\, r^2 \mathcal{F}_n^\ell(2,\beta r)\int d\Omega\, Y_\ell^m(\Omega)^*\Phi(\mathbf{r}). \tag{10}$$

Suppose we had chosen to expand j_{ℓ_1} and j_ℓ in terms of MHO functions instead of expanding j_{ℓ_1} and j_{ℓ_2}. Then in Eq. (5a) $\mathcal{F}^{\ell_2}_{n_2}(\alpha_2,\ \beta_2 r_2)$ would be replaced by $\mathcal{F}^{\ell}_{n}(\alpha,\beta r)$, and $\mathcal{G}^{\ell\ell_1\ell_2}_{n_1 n_2}(r)$ would be replaced by

$$\mathcal{G}^{\ell_2\ell_1\ell}_{n_1 n}(r_2)=\int_0^\infty dk\,k^2 j_{\ell_2}(kr_2)\,\mathcal{C}^{\ell_1}_{n_1}\left(\alpha_1,\frac{k}{\beta_1}\right)\mathcal{C}^{\ell}_{n}\left(\alpha,\frac{k}{\beta}\right). \tag{5c}$$

Evaluating Eq. (5c) as we did Eq. (5b) would then lead to the following alternative form for the expansion:

$$\Phi(\mathbf{r}_1-\mathbf{r}_2)$$
$$=\sum_{\substack{\ell m n s\\ \ell_1 m_1 n_1 s_1\\ \ell_2 m_2}} Y_{\ell_1}^{m_1}(\Omega_1)Y_{\ell_2}^{m}(\Omega_2)^*\mathcal{F}_{n_1}^{\ell_1}(\alpha_1,\beta_1 r_1)\mathcal{F}_{n_2}^{\ell_2}(2,\beta_2 r_2)$$
$$\times \begin{Bmatrix}\ell_1 m_1\\ \ell_2 m_2\\ \ell m\end{Bmatrix}\begin{Bmatrix} sn\ell\beta\alpha\\ s_1 n_1\ell_1\beta_1\alpha_1\\ n_2\ell_2\beta_2\end{Bmatrix}$$
$$\times \int dr\, r^2\mathcal{F}^\ell_n(\alpha,\beta r)\int d\Omega\, Y_\ell^m(\Omega)^*\Phi(\mathbf{r}), \tag{11a}$$

where now

$$n_2 = s+s_1+\tfrac{1}{2}(\ell+\ell_1-\ell_2), \tag{11b}$$

$$\beta_2=\left\{\frac{\beta^2(2-\alpha)\beta_1^2(2-\alpha_1)}{2[\beta^2(2-\alpha)+\beta_1^2(2-\alpha_1)]}\right\}^{\frac{1}{2}}. \tag{11c}$$

Equation (11a) is the expansion of $\Phi(\mathbf{r}_1-\mathbf{r}_2)$ which we will apply to the calculation of the DWBA nucleon-transfer-reaction amplitude. Note that

$$\begin{Bmatrix} sn\ell\beta\alpha\\ s_1 n_1\ell_1\beta_1\alpha_1\\ n_2\ell_2\beta_2\end{Bmatrix}=4\beta_2^3\left(\frac{\pi}{2}\right)^{\frac{3}{2}}$$
$$\times\frac{[n_2!\,\Gamma(n_2+\ell_2+\frac{3}{2})n!\,\Gamma(n+\ell+\frac{3}{2})n_1!\,\Gamma(n_1+\ell_1+\frac{3}{2})]^{\frac{1}{2}}}{s_1!\,(n_1-s_1)!\,\Gamma(s_1+\ell_1+\frac{3}{2})s\,!(n-s)!\,\Gamma(s+\ell+\frac{3}{2})}$$
$$\times\frac{(2\beta_2)^{2s_1+2s+\ell_1+\ell}}{\beta_1^{2s_1+\ell_1}\beta^{2s+\ell}}\times\frac{(-\alpha_1)^{n_1-s_1}}{(2-\alpha_1)^{s_1+n_1+\ell_1+\frac{3}{2}}}\times\frac{(-\alpha)^{n-s}}{(2-\alpha)^{s+n+\ell_1+\frac{3}{2}}} \tag{12}$$

and

$$\begin{Bmatrix} sn\ell\beta 0\\ s_1 n_1\ell_1\beta_1 0\\ n_2\ell_2\beta_2\end{Bmatrix}=\delta_{n_1,s_1}\delta_{n,s}\begin{Bmatrix} nn\ell\beta 0\\ n_1 n_1\ell_1\beta_1 0\\ n_2\ell_2\beta_2\end{Bmatrix}. \tag{13}$$

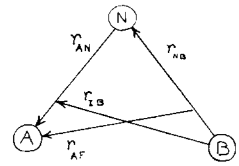


FIG. 1. Coordinate vectors for the nucleon transfer reaction $I+B=(A+N)+B\to A+(B+N)=A+F$.

III. NUCLEON-TRANSFER AMPLITUDE (NO RECOIL)

We return to Eq. (1) for the reaction amplitude. In Fig. 1 the various coordinate vectors are displayed. We will use $\mathbf{r}_{IB}$ and $\mathbf{r}_{NB}$ as our independent variables. Then $\mathbf{r}_{AF}$ and $\mathbf{r}_{AN}$ will be given by

$$\mathbf{r}_{AF}=\frac{M_A+M_N}{M_A}\mathbf{r}_{IB}-\frac{M_N}{M_A}\left(\frac{M_B+M_N+M_A}{M_B+M_N}\right)\mathbf{r}_{NB}, \tag{14}$$

$$\mathbf{r}_{AN}=\frac{M_N+M_A}{M_A}(\mathbf{r}_{IB}-\mathbf{r}_{NB}), \tag{15}$$

where M_X is the mass of particle X. Writing Eq. (1) in terms of the independent variables gives

$$A_{IF}=\left(\frac{M_I}{M_A}\right)^3\int d^3r_{NB}\int d^3r_{IB}\Phi^{(-)}_{AF}$$
$$\times\left(\mathbf{K}_A,\frac{M_I}{M_A}\mathbf{r}_{IB}-\frac{M_N}{M_A}\left(1+\frac{M_A}{M_F}\right)\mathbf{r}_{NB}\right)^*$$
$$\times\,\phi_{NB}(\mathbf{r}_{NB})^*V_{AN}\left(\frac{M_I}{M_A}(\mathbf{r}_{IB}-\mathbf{r}_{NB})\right)$$
$$\times\,\phi_{AN}\left(\frac{M_I}{M_A}(\mathbf{r}_{IB}-\mathbf{r}_{NB})\right)\Phi^{(+)}_{IB}(\mathbf{K}_I,\mathbf{r}_{IB}), \tag{16a}$$

where

$$M_I=M_N+M_A,\quad M_F=M_N+M_B. \tag{16b}$$

We first consider the case where $M_N\ll M_A, M_B$. Then we can neglect the recoil of A and set $\mathbf{r}_{AF}\approx(M_I/M_A)\mathbf{r}_{IB}\approx(M_B/M_F)\mathbf{r}_{IB}$. Now we use Eq. (11a) to expand $V_{AN}\phi_{AN}$. These two steps lead to the following expression for the reaction amplitude:

$$A_{IF}=\sum_{l_2 m_2 n_2}\mathcal{A}^{\ell_2 m_2 n_2}_{IF}(\mathbf{K}_A,\ \mathbf{K}_1,\beta_2)\mathcal{J}^{\ell_2 m_2 n_2}(\alpha,\beta;\alpha_1,\beta_1), \tag{17a}$$

where

$$\mathcal{A}^{\ell_2 m_2 n_2}_{IF}(\mathbf{K}_A,\mathbf{K}_I,\beta_2)$$
$$=\int d^3r\Phi^{(-)}_{AF}\left(\mathbf{K}_A,\frac{M_B}{M_F}\mathbf{r}\right)^*\mathcal{F}^{\ell_2}_{n_2}(2,\beta_2 r)Y^m_{l_2}(\Omega)^*$$
$$\times\,\Phi^{(+)}_{IB}(\mathbf{K}_I,\mathbf{r}), \tag{17b}$$

$$\mathfrak{Z}^{\ell_2 m_2 n_2}(\alpha, \beta; \alpha_1, \beta_1)$$

$$= \sum_{\substack{\ell_1 m_1 \ell m \\ s_1 n_1 n}} \mathfrak{Z}^{AN}_{\ell mn}(\alpha, \beta)\mathfrak{Z}^{NB}_{\ell_1 m_1 n_1}(\alpha_1, \beta_1)^* \begin{Bmatrix} \ell_1 m_1 \\ \ell_2 m_2 \\ \ell m \end{Bmatrix} \left(\frac{M_I}{M_A}\right)^3 \times \begin{Bmatrix} sn\ell\beta\alpha \\ s_1 n_1 \ell_1 \beta_1 \alpha_1 \\ n_2 \ell_2 \beta_2 \end{Bmatrix}, \quad (17c)$$

$$\beta_2 = \left\{\frac{\beta^2(2-\alpha)\beta_1^2(2-\alpha_1)}{2[\beta^2(2-\alpha) + \beta_1^2(2-\alpha_1)]}\right\}^{\frac{1}{2}}, \quad (17d)$$

$$s = n_2 - s_1 - \tfrac{1}{2}(\ell + \ell_1 - \ell_2), \quad (17e)$$

$$\mathfrak{Z}^{AN}_{\ell mn}(\alpha, \beta) = \int d^3r\, V_{AN}\left(\frac{M_I}{M_A} r\right)\phi_{AN}\left(\frac{M_I}{M_A} r\right) Y_\ell^m(\Omega)^* \mathcal{F}_n^\ell(\alpha, \beta r), \quad (17f)$$

$$\mathfrak{Z}^{NB}_{\ell_1 m_1 n_1}(\alpha_1, \beta_1) = \int d^3r\, \phi_{NB}(r) Y_{\ell_1}^{m_1}(\Omega)^* \mathcal{F}_{n_1}^{\ell_1}(\alpha_1, \beta_1 r). \quad (17g)$$

The quantity $\mathcal{A}_{IF}^{\ell_2 m_2 n_2}$ is just the sort of integral we get when we make the zero-range approximation. So the evaluation of the $\mathcal{A}_{IF}^{\ell_2 m_2 n_2}$'s presents no problem. The integrals $\mathfrak{Z}^{NA}$ and $\mathfrak{Z}^{NB}$ can be reduced to a sum of one-dimensional radial integrals once we have a spherical harmonic expansion of $V_{NA}\phi_{NA}$ and ϕ_{NB}. Thus the evaluation of the individual factors in our expression can be readily performed.

The usefulness of our method hinges on the range of values of n, n_1, and n_2 we must include in our sum to get an adequate representation. Clearly, if $V_{NA}\phi_{NA}$ and ϕ_{NB} are modified harmonic oscillator functions

$$V_{NA}\phi_{NA} = Y_{\ell'}^{m'}\mathcal{F}_{n'}^{\ell'}(2-\alpha, \beta r),$$

$$\phi_{NB} = Y_{\ell_1'}^{m_1'}\mathcal{F}_{n_1'}^{\ell_1'}(2-\alpha_1, \beta_1 r),$$

then there will be an upper limit of $n_1' + n' + \frac{1}{2}(l_1' + l' - l_2)$ on the sum over n_2, while the sum over n and n_1 is restricted to one term. In general, we expect that the values of α, β and α_1, β_1 can be chosen so that $\mathfrak{Z}^{NA}$ and $\mathfrak{Z}^{NB}$ will be nonnegligible only for a few values of n and n_1. Again, the sum over n_2 can have no more than $[n_1 + n + \frac{1}{2}(l_1 + l - l_2)]$ terms. For the choice $\alpha = \alpha_1 = 0$ there will be a unique value of n_2 corresponding to each pair n, n_1. We conclude that this expansion method appears to be a practical method for carrying out finite-range DWBA calculations for nucleon-transfer reactions.

IV. RECOIL EFFECTS

In neglecting recoil we replaced $\mathbf{r}_{AF}$ by $(M_B/M_F)\mathbf{r}_{IB}$ in the reaction amplitude A_{IF}. We chose $(M_B/M_F)\mathbf{r}_{IB}$ instead of $(M_I/M_A)\mathbf{r}_{IB}$ or just $\mathbf{r}_{IB}$, which are all equal to within terms of order M_N/M_A, so that our expression for A_{IF} reduces to the zero-range result in the limit as the range of $V_{AN}\phi_{AN}$ becomes small. For this reason we might expect that our results for A_{IF} will be pretty good even when M_N/M_A is not extremely small if the range of $V_{NA}\phi_{NA}$ is not so large and we approach the zero-range regime.

Dodd and Greider[4] have suggested that one can retain some measure of the recoil effect by using the substitution

$$\Phi_{AF}^{(-)}(\mathbf{K}_A, \mathbf{r}_{AF}) \approx \Phi_{AF}^{(-)}(\mathbf{K}_A, (M_B/M_B)\mathbf{r}_{IB}) \times \exp\{i\mathbf{K}_A \cdot (\mathbf{r}_{IB} - \mathbf{r}_{NB})[M_N(M_F + M_A)/M_A M_F]\}. \quad (18)$$

Inserting this into our formalism leads only to a change in the definition of $\mathfrak{Z}^{AN}$:

$$\mathfrak{Z}^{AN}_{\ell mn}(\alpha, \beta) = \int d^3r\, V_{AN}\left(\frac{M_I}{M_A}\mathbf{r}\right)\phi_{AN}\left(\frac{M_I}{M_A}\mathbf{r}\right) \times \exp\left[i\frac{M_N(M_F + M_A)}{M_A M_F}\mathbf{K}_A \cdot \mathbf{r}\right] \times Y_\ell^m(\Omega)^* \mathcal{F}_n^\ell(\alpha, \beta r). \quad (19)$$

For most cases the $\mathfrak{Z}^{AN}_{\ell mn}$ given by Eq. (17f) is nonvanishing only for a small range of l values. The $\mathfrak{Z}^{AN}_{\ell mn}$ given by Eq. (19), on the other hand, will allow a considerable spread in ℓ values, which can have an important effect on the final result.

Cases where both finite-range effects and recoil effects must be handled carefully lead to a much more complicated expression. For such cases we take $\mathbf{r}_{AF}$ and $\mathbf{r}_{IB}$ to be our independent variables. Then we use

$$\mathbf{r}_{NB} = \lambda_A(\mu_A \mathbf{r}_{IB} - \mathbf{r}_{AF}), \quad (20a)$$

$$\mathbf{r}_{AN} = \lambda_B(\mu_B \mathbf{r}_{AF} - \mathbf{r}_{IB}), \quad (20b)$$

$$\lambda_A = \frac{M_A M_F}{M_F M_I - M_B M_A}, \quad \mu_A = \frac{M_I}{M_A}, \quad (20c)$$

$$\lambda_B = \frac{M_B M_I}{M_I M_F - M_A M_B}, \quad \mu_B = \frac{M_F}{M_B}. \quad (20d)$$

In terms of these variables, the reaction amplitude becomes

$$A_{IF} = \lambda^3\mu^3 \int d^3\mathbf{r}_{AF} \int d^3\mathbf{r}_{IB} \Phi_{AF}^{(-)}(\mathbf{K}_A, \mathbf{r}_{AF})^* \times \phi_{NB}(\lambda_A(\mu_A \mathbf{r}_{IB} - \mathbf{r}_{AF})) \times V_{AN}\phi_{AN}(\lambda_B(\mu_B \mathbf{r}_{AF} - \mathbf{r}_{IB}))\Phi_{IB}^{(+)}(\mathbf{K}_I, \mathbf{r}_{IB}). \quad (21)$$

Now we use Eq. (11a) to expand both ϕ_{NB} and

$V_{NA}\phi_{NA}$. The result is

$$A_{IF} = \lambda_A^3 \mu_A^3 \sum_{\substack{\ell mns \ell_1 m_1 n_1 s_1 \ell_2 m_2 \\ \bar\ell \bar m \bar n \bar s \bar\ell_2 \bar m_2 \bar n_2 \bar s_2 \bar\ell_1 \bar m_1}} \mathfrak{I}^{\bar\ell_1 \bar m_1 \bar n_1}_{\ell_2 m_2 n_2} \mathfrak{F}^{\ell_1 m_1 n_1}_{\bar\ell_2 \bar m_2 \bar n_2} \mathfrak{Z}^{AN}_{\ell mn} \mathfrak{Z}^{NB*}_{\bar\ell \bar m \bar n}$$
$$\times \begin{Bmatrix} \ell_1 m_1 \\ \ell_2 m_2 \\ \ell m \end{Bmatrix} \begin{Bmatrix} \bar\ell_1 \bar m_1 \\ \bar\ell_2 \bar m_2 \\ \bar\ell \bar m \end{Bmatrix} \begin{Bmatrix} sn\ell\beta\alpha \\ s_1 n_1 \ell_1 \beta_1 \alpha_1 \\ n_2 \ell_2 \beta_2 \end{Bmatrix} \begin{Bmatrix} \bar s \bar n \bar\ell \bar\beta \bar\alpha \\ \bar s_2 \bar n_2 \bar\ell_2 \bar\beta_2 \bar\alpha_2 \\ \bar n_1 \bar\ell_1 \bar\beta_1 \end{Bmatrix}, \quad (22a)$$

where

$$\mathfrak{I}^{\bar\ell_1 \bar m_1 \bar n_1}_{\ell_2 m_2 n_2} = \int d^3\mathbf{r}\, \Phi^{(+)}_{IB}(\mathbf{K}_I, \mathbf{r}) Y^{m_2 *}_{\ell_2} Y^{\bar m_1 *}_{\bar\ell_1} \mathfrak{F}^{\bar\ell_1}_{\bar n_1}(\bar\alpha_1, \bar\beta_1 \mu_A r) \times \mathfrak{F}^{\ell_2}_{n_2}(2, \beta_2 r), \quad (22b)$$

$$\mathfrak{F}^{\ell_1 m_1 n_1}_{\bar\ell_2 \bar m_2 \bar n_2} = \int d^3\mathbf{r}\, \Phi^{(-)}_{AF}(\mathbf{K}_B, \mathbf{r})^* Y^{m_1}_{\ell_1} Y^{\bar m_2}_{\bar\ell_2} \mathfrak{F}^{\ell_1}_{n_1}(\alpha_1, \beta_1 \mu_B r) \times \mathfrak{F}^{\bar\ell_2}_{\bar n_2}(2, \bar\beta_2 r), \quad (22c)$$

$$\mathfrak{Z}^{AN}_{\ell mn} = \int d^3\mathbf{r}\, V_{AN}(\lambda_B \mathbf{r}) \phi_{AN}(\lambda_B \mathbf{r}) Y^{m*}_{\ell} \mathfrak{F}^{\ell}_{n}(\alpha, \beta r), \quad (22d)$$

$$\mathfrak{Z}^{NB}_{\bar\ell \bar m \bar n} = \int d^3\mathbf{r}\, \phi_{NB}(\lambda_A \mathbf{r}) Y^{\bar m *}_{\bar\ell} \mathfrak{F}^{\bar\ell}_{\bar m}(\bar\alpha, \bar\beta r), \quad (22e)$$

$$n_2 = s + s_1 + \tfrac{1}{2}(\ell + \ell_1 - \ell_2),$$
$$\beta_2 = \left\{ \frac{\beta^2(2-\alpha)\beta_1^2(2-\alpha_1)}{2[\beta^2(2-\alpha) + \beta_1^2(2-\alpha_1)]} \right\}^{\frac{1}{2}}, \quad (22f)$$
$$\bar n_1 = \bar s + \bar s_2 + \tfrac{1}{2}(\bar\ell + \bar\ell_2 - \bar\ell_1),$$
$$\bar\beta_1 = \left\{ \frac{\bar\beta^2(2-\bar\alpha)\bar\beta_2^2(2-\bar\alpha_2)}{2[\bar\beta^2(2-\bar\alpha) + \bar\beta_2^2(2-\bar\alpha_2)]} \right\}^{\frac{1}{2}}. \quad (22g)$$

Again we see that all the integrals reduce to sums of quadratures. The most apparent drawback of this method is the large number of terms that need to be included in the l sums.

V. THE ZERO-RANGE CASE

It is instructive to examine the limiting forms our expressions assume in the zero-range case. Applying Eq. (11a) to a δ function, we find

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{\ell,m,n} Y^m_\ell(\Omega_1) Y^m_\ell(\Omega_2)^* \mathfrak{F}^\ell_n(\alpha, \beta r_1) \times \mathfrak{F}^\ell_n(2-\alpha, \beta r_2)\beta^3. \quad (23)$$

In Eq. (16a) for the transition amplitude we set $V_{NA}(r)\phi_{NA}(r) = V_0\delta(r)$ and use Eq. (23). The no-recoil result is

$$A_{IF} = V_0\beta^3 \sum_{\ell,m,n} \int d^3\mathbf{R}\, \Phi^{(-)}_{AF}\left(\mathbf{K}_A, \frac{M_B}{M_F}\mathbf{R}\right) \times \mathfrak{F}^\ell_n(2-\alpha, \beta R) Y^m_\ell(\Omega)^* \times \Phi^{(+)}_{IB}(\mathbf{K}_I, \mathbf{R}) \int d^3r\, \phi_{NB}(\mathbf{r}) \mathfrak{F}^\ell_n(\alpha, \beta r) Y^m_\ell(\omega). \quad (24)$$

Alternatively,

$$A_{IF} = V_0 \int d^3R\, \Phi^{(-)}_{AF}\left(\mathbf{K}_A, \frac{M_B}{M_F}\mathbf{R}\right) \phi_{NB}(\mathbf{R}) \Phi^{(+)}_{IB}(\mathbf{K}_I, \mathbf{R}). \quad (25)$$

We see that even in the zero-range limit our expression for the reaction amplitude retains an expanded form, albeit considerably simplified from the finite-range result. This expansion can be reduced to one term if ϕ_{NB} is a modified harmonic-oscillator function. If ϕ_{NB} has a strong overlap with an MHO function, then the sum can be well represented by just a few terms.

APPENDIX A: THE BIORTHOGONAL PROPERTY OF THE MODIFIED HARMONIC-OSCILLATOR FUNCTION

The modified harmonic-oscillator (MHO) function is defined to be

$$\mathfrak{F}^\ell_n(\alpha, \rho) = \left[\frac{2\Gamma(n + \ell + \frac{3}{2})}{n!\Gamma(\ell + \frac{3}{2})^2} \right]^{\frac{1}{2}} \rho^\ell e^{-\frac{1}{2}\alpha\rho^2} \times {}_1F_1(-n; \ell + \tfrac{3}{2}; \rho^2), \quad (A1)$$

where $\Gamma(x)$ is the gamma function

$$\Gamma(x) = (x-1)\Gamma(x-1), \quad (A2)$$

and ${}_1F_1$ is the hypergeometric function

$${}_1F_1(a; b; x) = 1 + \frac{ax}{b1!} + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)x^3}{b(b+1)(b+2)3!} + \cdots = \sum_{n=0} \frac{\Gamma(a+N)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+n)} \frac{x^n}{n!}. \quad (A3)$$

$\mathfrak{F}^\ell_n(1, \rho)$ is the normalized radial harmonic-oscillator function.

The properties of the MHO functions will be investigated with the help of the auxiliary function

$$F^\ell_n(\alpha, \rho) = \frac{\Gamma(n + \ell + \frac{3}{2})}{\Gamma(\ell + \frac{3}{2})} (-\alpha)^n \rho^\ell e^{-\frac{1}{2}\rho^2} \times {}_1F_1(-n; \ell + \tfrac{3}{2}; \rho^2/\alpha) = (-1)^n \alpha^{n+\ell/2} [\tfrac{1}{2}\Gamma(n + \ell + \tfrac{3}{2}) n!]^{\frac{1}{2}} \mathfrak{F}^\ell_n(\alpha, \rho/\alpha^{\frac{1}{2}}) \quad (A4)$$

and the coefficient

$$N^\ell_{nn'}(\alpha) = \frac{\Gamma(n + \ell + \frac{3}{2})}{\Gamma(n' + \ell + \frac{3}{2})} \frac{n!}{n'!} \frac{(-\alpha)^{n-n'}}{(n-n')!}. \quad (A5)$$

The N coefficients satisfy a simple composition law

$$\sum_{b=c}^{a} N^\ell_{ab}(\alpha - \beta) N^\ell_{bc}(\beta) = N^\ell_{ac}(\alpha). \quad (A6)$$

The F function becomes a simple Gaussian times a power when α goes to zero:

$$F^{\ell}_{n}(0, \rho) = \rho^{2n+\ell} e^{-\rho^2/2}. \tag{A7}$$

It follows that every F function can be written as a sum of $F^{\ell}_{n}(0, \rho)$'s:

$$F^{\ell}_{n}(\alpha, \rho) = \sum_{s=0}^{n} N^{\ell}_{ns}(\alpha) F_{s}(0, \rho). \tag{A8}$$

The composition law for the N coefficients leads to a generalization of this result:

$$F^{\ell}_{n}(\alpha, \rho) = \sum_{r=0}^{n} N^{\ell}_{nr}(\alpha - \beta) F^{\ell}_{r}(\beta, \rho). \tag{A9}$$

Combining Eqs. (A4), (A8), and (A7), we find the following convenient representation of the MHO function:

$$\begin{aligned} \mathcal{F}^{\ell}_{n}(\alpha, \rho) &= \left(\frac{2}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right)^{\frac{1}{2}} \frac{(-1)^n}{(\alpha)^{n+\ell/2}} F^{\ell}_{n}(\alpha, \alpha^{\frac{1}{2}}\rho) \\ &= (-1)^n \left(\frac{2}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right)^{\frac{1}{2}} \\ &\quad \times \sum_{s=0}^{n} N^{\ell}_{ns}(\alpha) \alpha^{s-n} \rho^{2s+\ell} e^{-\frac{1}{2}\alpha\rho^2}. \end{aligned} \tag{A10}$$

Now we prove the biorthogonality of the MHO functions. Consider the integral

$$\begin{aligned} &\langle n \mid n' \rangle \\ &= \int_0^\infty d\rho\, \rho^2 \mathcal{F}^{\ell}_{n}(\alpha, \rho) \mathcal{F}^{\ell}_{n'}(2 - \alpha, \rho) \\ &= (-1)^{n+n'} 2 \sum_{s=0}^{n} \sum_{s'=0}^{n} N^{\ell}_{ns}(\alpha) N^{\ell}_{n's'}(-\alpha) \alpha^{s-n} (2 - \alpha)^{s'-n'} Q \\ &\quad \times [n!\,n'!\,\Gamma(n + \ell + \tfrac{3}{2}) \Gamma(n' + \ell + \tfrac{3}{2})]^{-\frac{1}{2}}, \end{aligned} \tag{A11}$$

where

$$Q = \int_0^\infty d\rho\, \rho^{2+2s+2s'+2\ell} e^{-\rho^2} = \tfrac{1}{2}\Gamma(s + s' + \ell + \tfrac{3}{2}).$$

Thus

$$\langle n \mid n' \rangle = \sum_{s=0}^{n} \sum_{s'=0}^{n'} \frac{N^{\ell}_{ns}(1) N^{\ell}_{n's'}(1) \Gamma(s + s' + \ell + \frac{3}{2})(-1)^{n+n'}}{[n!\,n'!\,\Gamma(n + \ell + \frac{3}{2}) \Gamma(n' + \ell + \frac{3}{2})]^{\frac{1}{2}}}.$$

Using Eq. (A5),

$$\begin{aligned} &\langle n \mid n' \rangle \\ &= \sum_{s=0}^{n} N_{ns\ell}(1) \left(\sum_{s'=0}^{n'} \frac{n'!\,\Gamma(n' + \ell + \frac{3}{2})(-1)^{n'-s'} \times \Gamma(s + s' + \ell + \frac{3}{2})(-1)^{n+n'}}{s'!\,\Gamma(s' + \ell + \frac{3}{2})(n' - s')!} \right) \\ &\quad \times [n!\,n'!\,\Gamma(n + \ell + \tfrac{3}{2}) \Gamma(n' + \ell + \tfrac{3}{2})]^{-\frac{1}{2}}. \end{aligned} \tag{A12}$$

But $n!/(n - s)! = (-1)^s \Gamma(s - n)/\Gamma(-n)$, so we have

$$\begin{aligned} \langle n \mid n' \rangle &= \left[\frac{n'!\,\Gamma(n' + \ell + \frac{3}{2})}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right]^{\frac{1}{2}} \frac{(-1)^n}{n'!} \\ &\quad \times \sum_{s=0}^{n} N^{\ell}_{ns}(1) \sum_{s'=0}^{n'} \frac{\Gamma(s + s' + \ell + \frac{3}{2})}{\Gamma(s' + \ell + \frac{3}{2})} \\ &\quad \times \frac{\Gamma(s' - n')}{\Gamma(-n')} \frac{(+1)^{s'}}{s'!}. \end{aligned}$$

Compare this with the hypergeometric function

$$_2F_1(\alpha, \beta; \gamma; \delta) = \sum_{s'} \frac{\Gamma(\alpha + s')}{\Gamma(\alpha)} \frac{\Gamma(\beta + s')}{\Gamma(\beta)} \frac{\Gamma(\gamma)}{\Gamma(\gamma + s')} \frac{\delta^{s'}}{s'!}. \tag{A13}$$

We see that

$$\begin{aligned} \langle n \mid n' \rangle &= \left[\frac{n'!\,\Gamma(n' + \ell + \frac{3}{2})}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right]^{\frac{1}{2}} \frac{(-1)^n}{n'!} \\ &\quad \times \sum_{s=0}^{n} N^{\ell}_{ns}(1) \frac{\Gamma(s + \ell + \frac{3}{2})}{\Gamma(\ell + \frac{3}{2})} \\ &\quad \times {}_2F_1(-n', s + \ell + \tfrac{3}{2}; \ell + \tfrac{3}{2}; +1). \end{aligned} \tag{A14}$$

Now we use the identity

$$\begin{aligned} &_2F_1(-a, b; c; \delta) \\ &= (1 - \delta)^a {}_2F_1(-a, c - b; c; \delta/[\delta - 1]) \\ &= \sum_{s=0}^{a} \frac{\Gamma(s - a)}{\Gamma(-a)} \frac{\Gamma(s + c - b)}{\Gamma(c - b)} \frac{\Gamma(c)}{\Gamma(s + c)} \frac{(-\delta)^s}{s!} (1 - \delta)^{a-s} \\ &\xrightarrow[\delta\to 1]{} \frac{\Gamma(a + c - b)}{\Gamma(c - b)} \frac{\Gamma(c)}{\Gamma(a + c)}. \end{aligned} \tag{A15}$$

Thus

$$\begin{aligned} \langle n \mid n' \rangle &= \left[\frac{n'!\,\Gamma(n' + \ell + \frac{3}{2})}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right]^{\frac{1}{2}} \\ &\quad \times \sum N^{\ell}_{ns}(1) \frac{\Gamma(s + \ell + \frac{3}{2})}{\Gamma(n' + \ell + \frac{3}{2})} \frac{s!}{n'!\,(s - n')!} \\ &= \left[\frac{n'!\,\Gamma(n' + \ell + \frac{3}{2})}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right]^{\frac{1}{2}} \sum_{s=0}^{n} N^{\ell}_{ns}(1) N^{\ell}_{sn'}(-1) \\ &= \left[\frac{n'!\,\Gamma(n' + \ell + \frac{3}{2})}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right]^{\frac{1}{2}} N^{\ell}_{nn'}(0) = \delta_{n'n}. \end{aligned} \tag{A16}$$

APPENDIX B: EXPANSION COEFFICIENT FOR THE SPHERICAL BESSEL FUNCTION

Our purpose here is to evaluate the coefficient

$$\mathcal{C}^{\ell}_{n}(2 - \alpha, \bar{n}) = \int_0^\infty d\rho\, \rho^2 j_\ell(\bar{n}\rho) \mathcal{F}^{\ell}_{n}(\alpha, \rho). \tag{B1}$$

From Eq. (A10) this can be transformed to

$$\begin{aligned} \mathcal{C}^{\ell}_{n}(2 - \alpha, \bar{n}) &= (-1)^n \left[\frac{2}{n!\,\Gamma(n + \ell + \frac{3}{2})}\right]^{\frac{1}{2}} \\ &\quad \times \sum_{s=0}^{n} N^{\ell}_{ns}(\alpha) \alpha^{s-n} \mathfrak{J}^{\ell}_{s}(\bar{n}, \alpha), \end{aligned} \tag{B2}$$

where N_{ns} is defined in Eq. (A5) and

$$\mathfrak{J}_s(\bar{n}, \alpha) = \int_0^\infty d\rho\, \rho^{2+2s+\ell} e^{-\frac{1}{2}\alpha\rho^2} j_\ell(\bar{n}\rho). \tag{B3}$$

We insert the Taylor's series for j_ℓ and integrate term by term.

$$\mathfrak{J}_s^\ell(\bar{n}, \alpha) = (2\bar{n})^\ell \sum_{m=0}^\infty \frac{(-1)^m(\ell + m)!\, \bar{n}^{2m}}{m!(2 \quad + 2m + 1)!} \times \frac{1 \cdot 3 \cdot 5 \cdots (1 + 2s + 2m + 2\ )}{2^{2+s+m+\ell}(\frac{1}{2}\alpha)^{\frac{3}{2}+s+m+\ell}}$$

$$= \frac{(2\bar{n})\, 2^{s-\ell}}{\alpha^{(\frac{3}{2}+s+\ )}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{m=0}^\infty \frac{\Gamma(\frac{3}{2} + s + m + \ell)}{\Gamma(\frac{3}{2} + m + \ell)} \frac{\left(-\dfrac{\bar{n}^2}{2\alpha}\right)^m}{m!}$$

$$= \frac{\bar{n}\, 2^s}{\alpha^{(\frac{3}{2}+s+\ell)}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{3}{2} + s + \ell)}{\Gamma(\frac{3}{2} + \ell)} \times {}_1F_1(\tfrac{3}{2} + s + \ell; \tfrac{3}{2} + \ell; -\bar{n}^2/2\alpha). \tag{B4}$$

Thus we find that $\mathfrak{J}_s^\ell$ is a hypergeometric function. We now employ the Kummer transformation

$${}_1F_1(a; c; x) = e^x {}_1F_1(c - a; c; -x)$$

to give

$$\mathfrak{J}_s^\ell(\bar{n}, \alpha) = \frac{\bar{n}^\ell 2^s}{\alpha^{\frac{3}{2}+s+\ell}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{3}{2} + s + \ell)}{\Gamma(\frac{3}{2} + \ell)} \times e^{-n^2/2\alpha} {}_1F_1(-s; \ell + \tfrac{3}{2}; \bar{n}^2/2\alpha). \tag{B5}$$

Comparison of this expression with the definition of F_n^ℓ in Eq. (A4) leads to

$$\mathfrak{J}_s^\ell(\bar{n}, \alpha) = \alpha^{-(\frac{3}{2}+\frac{1}{2}\ell)} (\tfrac{1}{2}\pi)^{\frac{1}{2}} (-\alpha)^{-s} F_s(2, \bar{n}/\alpha^{\frac{1}{2}}). \tag{B6}$$

This is now substituted back into Eq. (B2) and use is made of Eq. (A5):

$$\mathcal{C}_n^\ell(2 - \alpha, \bar{n}) = \frac{\pi^{\frac{1}{2}} \alpha^{-\frac{3}{2}-\frac{1}{2}\ell-n}}{[n!\,\Gamma(n + \pi + \frac{3}{2})]^{\frac{1}{2}}} \times \sum N_{ns}^\ell(-\alpha) F_s^\ell(2, \bar{n}/\alpha^{\frac{1}{2}}). \tag{B7}$$

We next make use of Eqs. (A9) and (A4):

$$\mathcal{C}_n^\ell(2 - \alpha, \bar{n}) = \frac{\pi^{\frac{1}{2}} \alpha^{-\frac{3}{2}-\frac{1}{2}\ell-n}}{[n!\,\Gamma(n + \ell + \frac{3}{2})]^{\frac{1}{2}}} F_n^\ell(2 - \alpha, \bar{n}/\alpha^{\frac{1}{2}})$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(2 - \alpha)^{n+\frac{1}{2}\ell}}{\alpha^{(\frac{3}{2}+n+\frac{1}{2}\ell)}} (-1)^n \times \mathcal{F}_n^\ell\left(2 - \alpha, \frac{\bar{n}}{[\alpha(2 - \alpha)]^{\frac{1}{2}}}\right). \tag{B8}$$

Thus, if we make a simple change of variables

$$\mathcal{C}_n^\ell\left(\alpha, \frac{k}{\beta}\right) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\alpha^{n+\frac{1}{2}\ell}}{(2 - \alpha)^{n+\frac{1}{2}+\frac{3}{2}}} (-1)^n \times \mathcal{F}_n^\ell\left(\alpha, \frac{k}{\beta[\alpha(2 - \alpha)]^{\frac{1}{2}}}\right). \tag{B9}$$

APPENDIX C: EVALUATION OF THE FUNCTION $\mathcal{G}_{n_1 n_2}^{\ell\ell_1\ell_2}$

The function $\mathcal{G}_{n_1 n_2}^{\ell\ell_1\ell_2}(r)$ is defined by

$$\mathcal{G}_{n_2 n_1}^{\ell\ell_1\ell_2}(r) = \int_0^\infty dk\, k^2 j_\ell(kr) \mathcal{C}_{n_1}^{\ell_1}\left(\alpha_1, \frac{k}{\beta_1}\right) \mathcal{C}_{n_2}^{\ell_2}\left(\alpha_2, \frac{k}{\beta_2}\right). \tag{C1}$$

We use Eq. (9) to replace the $\mathcal{C}$'s:

$$\mathcal{G}_{n_1 n_2}^{\ell\ell_1\ell_2}(r) = (-1)^{n_1+n_2} \frac{\pi}{2} \frac{\alpha_1^{(n_1+\frac{1}{2}\ell_1)} \alpha_2^{(n_2+\frac{1}{2}\ell_2)} g(r)}{(2 - \alpha_1)^{(n_1+\frac{1}{2}\ell_1+\frac{3}{2})} (2 - \alpha_2)^{(n_2+\frac{1}{2}\ell_2+\frac{3}{2})}}, \tag{C2a}$$

$$g(r) = \int_0^\infty dk\, k^2 j^\ell(kr) \mathcal{F}_{n_2}^{\ell_1}\left(\alpha_1, \frac{k}{\beta_1[\alpha_1(2 - \alpha_1)]^{\frac{1}{2}}}\right) \times \mathcal{F}_{n_2}^{\ell_2}\left(\alpha_2, \frac{k}{\beta_2[\alpha_2(2 - \alpha_2)]^{\frac{1}{2}}}\right). \tag{C2b}$$

Equation (A10) is used to replace the $\mathcal{F}$'s.

$$g(r) = \frac{(-1)^{n_1+n_2} 2 \left[\sum_{s_1=0}^{n_1} \sum_{s_2=0}^{n_2} N_{n_1 s_1}^{\ell_1}(\alpha_1) N_{n_2 s_2}^{\ell_2}(\alpha_2) \alpha_1^{s_1-n_1} \alpha_2^{s_2-n_2} h(r)\right]}{[n_1!\,\Gamma(n_1 + \ell_1 + \frac{3}{2}) n_2!\,\Gamma(n_2 + \ell_2 + \frac{3}{2})]^{1/2}},$$

$$h(r) = \left(\frac{1}{\beta_1[\alpha_1(2 - \alpha_1)]^{\frac{1}{2}}}\right)^{2s_1+\ell_1} \left(\frac{1}{\beta_2[\alpha_2(1 - \alpha_2)]}\right)^{2s_2+\ell_2} \tag{C3a}$$

$$\times \int_0^\infty dk e^{-k_2/4\beta_2} k^{2+2n+\ell} j_\ell(kr), \tag{C3b}$$

$$\beta = \left[\frac{\beta_1^2(2 - \alpha_1)\beta_2^2(2 - \alpha_2)}{2[\beta_1^2(2 - \alpha_1) + \beta_2^2(2 - \alpha_2)]}\right]^{\frac{1}{2}}, \tag{C3c}$$

$$n = s_1 + s_2 + \tfrac{1}{2}(\ell_1 + \ell_2 - \ell). \tag{C3d}$$

With the help of Eqs. (B3) and (B6) we find

$$h(r) = (\beta_1[\alpha_1(2 - \alpha_1)]\ )^{-2s_1-\ell_1} \times (\beta_2[\alpha_2(1 - \alpha_2)]^{\frac{1}{2}})^{-2s_2-\ell_2} \mathfrak{J}_n(r, 1/2\beta^2)$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (-1)^n \times \frac{(2\beta^2)^{n+\frac{1}{2}\ell+\frac{3}{2}} F_n^\ell(2, r\beta\sqrt{2})}{(\beta_1[\alpha_1(2 - \alpha_1)]^{\frac{1}{2}})^{2s_1+\ell_1} (\beta_2[\alpha_2(1 - \alpha_2)]^{\frac{1}{2}})^{2s_2+\ell_2}}. \tag{C4}$$

Now we use Eq. (A4)

$$h(r) = \frac{[2\pi n!\,\Gamma(n + \ell + \frac{3}{2})]^{\frac{1}{2}} \beta^3 (2\beta)^{2n+\ell} \mathcal{F}_n^\ell(2, \beta r)}{(\beta_1[\alpha_1(2 - \alpha_1)]^{\frac{1}{2}})^{2s_1+\ell_1} (\beta_2[\alpha_2(2 - \alpha_2)]^{\frac{1}{2}})^{2s_2+\ell_2}}. \tag{C5}$$

Substituting back into Eqs. (C2) and (C3)

$$\mathcal{G}^{\ell_1\ell_2}_{n_1n_2}(r) = \frac{\pi^{\frac{3}{2}}\alpha_1^{(n+\frac{1}{2}\ell_1)}\alpha_2^{(n_2+\frac{1}{2}\ell_2)}\sum_{s_1=0}^{n_1}\sum_{s_2=0}^{n_2} N^{\ell_1}_{n_1s_1}(\alpha_1)N^{\ell_2}_{n_2s_2}(\alpha_2)\alpha_1^{s_1-n_1}\alpha_2^{s_2-n}}{(2-\alpha_1)^{n_1+\frac{1}{2}\ell_1+\frac{3}{2}}(2-\alpha_2)^{n_2+\frac{1}{2}\ell_2+\frac{3}{2}}}$$

$$\times [n_1!\Gamma(n_1+\ell_1+\tfrac{3}{2})n_2!\Gamma(n_2+\ell_2+\tfrac{3}{2})]^{\frac{1}{2}}$$

$$\times \frac{2^{2n+\ell+\frac{1}{2}}[n!\Gamma(n+\ell+\frac{3}{2})]^{\frac{1}{2}}\beta^{2n+\ell+3}}{(\beta_1[\alpha_1(2-\alpha_1)]^{\frac{1}{2}})^{2s_1+\ell_1}(\beta_2[\alpha_2(2-\alpha_2)]^{\frac{1}{2}})^{2s_2+\ell_2}}\mathcal{F}_n(2,\beta r). \tag{C6}$$

Now eliminate the N's with Eq. (A5):

$$\mathcal{G}^{\ell\ell_1\ell_2}_{n_1n_2}(r) = \left(\frac{\pi}{2}\right)^{\frac{3}{2}}\sum_{s_1=0}^{n_1}\sum_{s_2=0}^{n_2}\frac{[n_1!\Gamma(n_1+\ell_1+\frac{3}{2})n_2!\Gamma(n_2+\ell_2+\frac{3}{2}) \times n!\Gamma(n+\ell+\frac{3}{2})]^{\frac{1}{2}}}{2s_1!\Gamma(s_1+\ell_1+\frac{3}{2})s_2!\Gamma(s_2+\ell_2+\frac{3}{2}) \times (n_1-s_1)!(n_2-s_2)!}$$

$$\times \frac{(-\alpha_1)^{n_1-s_1}(-\alpha_2)^{n_2-s_2}(2\beta)^{2n+\ell+3}\mathcal{F}_n(2,\beta r)}{(2-\alpha_1)^{n_1+s_2+\ell_1+\frac{3}{2}}(2-\alpha_1)^{n_2+s_2+\ell_2+\frac{3}{2}}\beta_1^{2s_1+\ell_1}\beta_2^{2s_1+\ell_1}}$$

$$= \sum_{s_1=0}^{n_1}\sum_{s_2=0}^{n_2}\begin{Bmatrix} s_1n_1\ell_1\beta_1\alpha_1 \\ s_2n_2\ell_2\beta_2\alpha_2 \\ n\ell\beta \end{Bmatrix}\mathcal{F}^{\ell}_n(2,\beta r). \tag{C7}$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Partial Group-Theoretical Treatment for the Relativistic Hydrogen Atom

H. BACRY*
The Institute for Advanced Study, Princeton, New Jersey
AND
J. L. RICHARD
Université de Marseille, Marseille, France

(Received 15 March 1967; final manuscript received 26 April 1967)

The Lie algebra of $SU(1,1)$ and its Hermitian representations are used together with spherical harmonics to solve the wave equations for the nonrelativistic q-dimensional oscillator and the relativistic Kepler problem.

INTRODUCTION

In the case of spherical potentials, the use of spherical harmonics reduces the solution of the wave equation to the solution of a radial equation corresponding to a formal one-dimensional problem. In some interesting cases, the radial equation can be put in the form

$$-\frac{d^2X}{dx^2}+(x^2+K/x^2)X = wX. \tag{1}$$

In other words, one has to find the eigenvalues w of the Hamiltonian

$$H = \pi^2 + x^2 + (K/x^2), \tag{2}$$

where

$$\pi = -i(d/dx) \tag{3}$$

It is shown in Sec. I that the whole spectrum of H can be found with the aid of a *noninvariance group*, namely the covering group $\overline{SU(1,1)}$ of the two-dimensional Lorentz group. The unitary representations of $SU(1,1)$ have been obtained by Bargmann,[1] those of $\overline{SU(1,1)}$ by Pukanszky.[2] Let us mention also the works of Sannikov[3] and Barut and Fronsdal.[4]

In Sec. II, a complete classification of the states of the nonrelativistic q-dimensional harmonic oscillator is given, using the spherical harmonics of $SO(q)$ and unitary representations of the group $\overline{SU(1,1)}$.

The case of the relativistic hydrogen atom is examined in Sec. III.

I. THE ONE-DIMENSIONAL PROBLEM

In this section, we intend to find the spectrum of the Hamiltonian (1) corresponding to a particle of

* On leave from Université de Marseille. Work supported by the National Science Foundation.

[1] V. Bargmann, Ann. Math. **48**, 568 (1947).
[2] L. Pukanszky, Math. Ann. **156**, 96 (1964).
[3] S. S. Sannikov, Zh. Eksperim. i Teor. Fiz. **49**, 1913 (1965) [English transl.: Soviet Phys.—JETP **22**, 1306 (1966)].
[4] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A287**, 532 (1965).

Substituting back into Eqs. (C2) and (C3)

$$\mathcal{G}^{\ell_1\ell_2}_{n_1n_2}(r) = \frac{\pi^{\frac{3}{2}}\alpha_1^{(n+\frac{1}{2}\ell_1)}\alpha_2^{(n_2+\frac{1}{2}\ell_2)}\sum_{s_1=0}^{n_1}\sum_{s_2=0}^{n_2} N^{\ell_1}_{n_1s_1}(\alpha_1)N^{\ell_2}_{n_2s_2}(\alpha_2)\alpha_1^{s_1-n_1}\alpha_2^{s_2-n}}{(2-\alpha_1)^{n_1+\frac{1}{2}\ell_1+\frac{3}{2}}(2-\alpha_2)^{n_2+\frac{1}{2}\ell_2+\frac{3}{2}}} \times [n_1!\,\Gamma(n_1+\ell_1+\tfrac{3}{2})n_2!\,\Gamma(n_2+\ell_2+\tfrac{3}{2})]^{\frac{1}{2}} \times \frac{2^{2n+\ell+\frac{1}{2}}[n!\,\Gamma(n+\ell+\frac{3}{2})]^{\frac{1}{2}}\beta^{2n+\ell+3}}{(\beta_1[\alpha_1(2-\alpha_1)]^{\frac{1}{2}})^{2s_1+\ell_1}(\beta_2[\alpha_2(2-\alpha_2)]^{\frac{1}{2}})^{2s_2+\ell_2}}\mathcal{F}_n(2,\beta r). \tag{C6}$$

Now eliminate the N's with Eq. (A5):

$$\mathcal{G}^{\ell\ell_1\ell_2}_{n_1n_2}(r) = \left(\frac{\pi}{2}\right)^{\frac{3}{2}}\sum_{s_1=0}^{n_1}\sum_{s_2=0}^{n_2}\frac{[n_1!\,\Gamma(n_1+\ell_1+\frac{3}{2})n_2!\,\Gamma(n_2+\ell_2+\frac{3}{2})\times n!\,\Gamma(n+\ell+\frac{3}{2})]^{\frac{1}{2}}}{2s_1!\,\Gamma(s_1+\ell_1+\frac{3}{2})s_2!\,\Gamma(s_2+\ell_2+\frac{3}{2})\times(n_1-s_1)!\,(n_2-s_2)!} \times\frac{(-\alpha_1)^{n_1-s_1}(-\alpha_2)^{n_2-s_2}(2\beta)^{2n+\ell+3}\mathcal{F}_n(2,\beta r)}{(2-\alpha_1)^{n_1+s_2+\ell_1+\frac{3}{2}}(2-\alpha_1)^{n_2+s_2+\ell_2+\frac{3}{2}}\beta_1^{2s_1+\ell_1}\beta_2^{2s_1+\ell_1}}$$

$$= \sum_{s_1=0}^{n_1}\sum_{s_2=0}^{n_2}\begin{Bmatrix} s_1n_1\ell_1\beta_1\alpha_1 \\ s_2n_2\ell_2\beta_2\alpha_2 \\ n\ell\beta \end{Bmatrix}\mathcal{F}^{\ell}_n(2,\beta r). \tag{C7}$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Partial Group-Theoretical Treatment for the Relativistic Hydrogen Atom

H. BACRY*
The Institute for Advanced Study, Princeton, New Jersey
AND
J. L. RICHARD
Université de Marseille, Marseille, France

(Received 15 March 1967; final manuscript received 26 April 1967)

The Lie algebra of $SU(1, 1)$ and its Hermitian representations are used together with spherical harmonics to solve the wave equations for the nonrelativistic q-dimensional oscillator and the relativistic Kepler problem.

INTRODUCTION

In the case of spherical potentials, the use of spherical harmonics reduces the solution of the wave equation to the solution of a radial equation corresponding to a formal one-dimensional problem. In some interesting cases, the radial equation can be put in the form

$$-\frac{d^2X}{dx^2} + (x^2 + K/x^2)X = wX. \tag{1}$$

In other words, one has to find the eigenvalues w of the Hamiltonian

$$H = \pi^2 + x^2 + (K/x^2), \tag{2}$$

where

$$\pi = -i(d/dx) \tag{3}$$

It is shown in Sec. I that the whole spectrum of H can be found with the aid of a *noninvariance group*, namely the covering group $\overline{SU(1, 1)}$ of the two-dimensional Lorentz group. The unitary representations of $SU(1, 1)$ have been obtained by Bargmann,[1] those of $\overline{SU(1, 1)}$ by Pukanszky.[2] Let us mention also the works of Sannikov[3] and Barut and Fronsdal.[4]

In Sec. II, a complete classification of the states of the nonrelativistic q-dimensional harmonic oscillator is given, using the spherical harmonics of $SO(q)$ and unitary representations of the group $\overline{SU(1, 1)}$.

The case of the relativistic hydrogen atom is examined in Sec. III.

I. THE ONE-DIMENSIONAL PROBLEM

In this section, we intend to find the spectrum of the Hamiltonian (1) corresponding to a particle of

* On leave from Université de Marseille. Work supported by the National Science Foundation.

[1] V. Bargmann, Ann. Math. **48**, 568 (1947).
[2] L. Pukanszky, Math. Ann. **156**, 96 (1964).
[3] S. S. Sannikov, Zh. Eksperim. i Teor. Fiz. **49**, 1913 (1965) [English transl.: Soviet Phys.—JETP **22**, 1306 (1966)].
[4] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. (London) **A287**, 532 (1965).

TABLE I. Unitary representations of $SU(1, 1)$.

Name	$D_P(Q, h)$ (Principal series)	$D_S(Q, h)$ (Supplementary series)	$D^+(\sigma)$	$D^-(\sigma)$	Trivial
Invariants	$\sigma = -\frac{1}{2} + is$, $s \in \mathbb{R}$, $-\frac{1}{2} < h \leq \frac{1}{2}$ (case $h = \frac{1}{2}$, $s = 0$ excluded)	$0 < \sigma < \frac{1}{2} - \lvert h \rvert$, $-\frac{1}{2} < h < \frac{1}{2}$	σ real negative		$\sigma = 0$
J_3 spectrum	$h + n'$, $n' = 0, \pm 1, \pm 2, \pm 3, \cdots$		$-\sigma, -\sigma + 1, -\sigma + 2, \cdots$	$\sigma, \sigma - 1, \sigma - 2, \cdots$	0

mass $\frac{1}{2}$ placed in the potential $V(x) = x^2 + (K/x^2)$, where K is an arbitrary constant.

The dynamical variables

$$J_1 = \tfrac{1}{4}(H - 2x^2), \tag{4}$$

$$J_2 = \tfrac{1}{4}(x\pi + \pi x), \tag{5}$$

$$J_3 = \tfrac{1}{4}H, \tag{6}$$

are clearly Hermitian. Using the property

$$[x, \pi] = i, \tag{7}$$

one readily derives the commutation relations

$$[J_1, J_2] = -iJ_3, \tag{8}$$

$$[J_2, J_3] = iJ_1, \tag{9}$$

$$[J_3, J_1] = iJ_2, \tag{10}$$

which are those of the Lie algebra of $SU(1, 1)$. Consequently, the Hermitian operators J_1, J_2, J_3 generate a unitary representation of $\overline{SU(1,1)}$. The computation of the Casimir operator Q leads to the constant

$$Q = -J_1^2 - J_2^2 + J_3^2 = \tfrac{1}{4}(K - \tfrac{3}{4}). \tag{11}$$

Note that the equation

$$Q = \sigma(\sigma + 1) \tag{12}$$

has two solutions, namely

$$\sigma_1 = -\tfrac{1}{2} - \tfrac{1}{2}[K + \tfrac{1}{4}]^{\frac{1}{2}}, \tag{13}$$

$$\sigma_2 = -\tfrac{1}{2} + \tfrac{1}{2}[K + \tfrac{1}{4}]^{\frac{1}{2}}. \tag{14}$$

As shown in references above, the number σ is generally not sufficient to characterize the unitary representation of $\overline{SU(1,1)}$. Another number h, the range of which is $[-\frac{1}{2}, \frac{1}{2}]$, is needed. This number is defined as the fractional part of the eigenvalues of J_3. Given a unitary representation of $\overline{SU(1,1)}$, the spectrum of J_3 is well defined without any degeneracy, as indicated in Table I. Note that the number h occurs only in the principal series $D_P(Q, h)$ and the supplementary series $D_S(Q, h)$ and that for each value of $\sigma \leq -1$ correspond only two unitary representations, one of the class $D^+(\sigma)$ with a lower bound for the J_3 spectrum, the other of the class $D^-(\sigma)$ with an upper bound.

Let us apply these properties to the case of the harmonic oscillator in one dimension:

$$-(\hbar^2/2m)(d^2R/dr^2) + \tfrac{1}{2}m\omega^2r^2R = ER. \tag{15}$$

By introducing the new variables

$$x = (m\omega/\hbar)^{\frac{1}{2}}r, \tag{16}$$

$$w = 2E/\hbar\omega, \tag{17}$$

we are led to Eq. (1) with $K = 0$. For this value of K, one recognizes the Lie algebra proposed by Sannikov.[3] According to (13) and (14), the values of σ are

$$\sigma_1 = -\tfrac{3}{4}, \tag{18}$$

$$\sigma_2 = -\tfrac{1}{4}. \tag{19}$$

They are both negative. Let us verify that the corresponding representations $D^+(-\frac{3}{4})$ and $D^+(-\frac{1}{4})$ provide us all energy levels. In fact, according to Table I and Eqs. (1), (2), and (6), we get

$$\tfrac{1}{4}w = -\sigma + n', \tag{20}$$

where n' takes the values $0, 1, 2, 3, \cdots$. Therefore from (17), the energy spectrum for E is given by

$$D^+(-\tfrac{3}{4}){:}\ E = (2n' + \tfrac{3}{2})\hbar\omega, \tag{21}$$

$$D^+(-\tfrac{1}{4}){:}\ E = (2n' + \tfrac{1}{2})\hbar\omega. \tag{22}$$

These relations prove that the levels are classified in a unitary *reducible* representations of $\overline{SU(1,1)}$, $D^+(-\frac{3}{4})$ containing all odd levels and $D^+(-\frac{1}{4})$ containing all even levels.

That the representations D^- and D_S have to be discarded can be understood from the positiveness of the energy.

As another one-dimensional example, consider the case of the Hamiltonian (2) with $K \leq -\frac{1}{4}$. According to (12), only the principal series can be used. This problem corresponds to an infinitely deep and infinitely high potential. A direct investigation[5] of the solutions proves that the spectrum is discrete but that the energy is defined up to an additive constant, a property which corresponds to the fact that a representation of $D_P(Q, h)$ is defined up to the choice of the value of h.

It is interesting now to look at some more physical examples.

II. THE q-DIMENSIONAL HARMONIC OSCILLATOR ($q > 1$)

One starts from the radial wave equation for the q-dimensional (nonrelativistic) harmonic oscillator

$$\left\{\frac{d^2}{dr^2} + \frac{q-1}{r}\frac{d}{dr} - \frac{l(l+q-2)}{r^2} - \frac{m^2\omega^2}{\hbar^2}r^2 + \frac{2mE}{\hbar^2}\right\}R(r) = 0, \quad (23)$$

where l is the degree of the spherical harmonics in the q-dimensional Euclidean space.

By the following change of variable and function:

$$x = (m\omega/\hbar)^{\frac{1}{2}}r, \quad (24)$$

$$X(x) = x^{(q-1/\hbar)}R, \quad (25)$$

Eq. (23) takes the form

$$\{-d^2/dx^2 + x^2 + K/x^2 - w\}X(x) = 0, \quad (26)$$

with

$$w = 2E/\hbar\omega, \quad (27)$$

$$K = l(l+q-2) + \tfrac{1}{4}(q-1)(q-3). \quad (28)$$

Putting this value of K in (13) and (14), one gets

$$\sigma_1 = -\tfrac{1}{2}l - \tfrac{1}{4}q, \quad (29)$$

$$\sigma_2 = \tfrac{1}{2}l + \tfrac{1}{4}q - 1. \quad (30)$$

The only representations which are compatible with the positiveness of the energy are those of the type $D^+(\sigma)$. The only value of σ which is acceptable is σ_1 which is negative for all values of l. The spectrum of J_3 for $D^+(-\frac{1}{2}l - \frac{1}{4}q)$ is

$$\tfrac{1}{4}w = \tfrac{1}{2}l + \tfrac{1}{4}q + n' \quad (n' = 0, 1, 2, 3, \cdots). \quad (31)$$

Equation (27) leads to the energy spectrum

$$E = (l + 2n' + \tfrac{1}{2}q)\hbar\omega = (n + \tfrac{1}{2}q)\hbar\omega, \quad (32)$$

where

$$n = l + 2n' \quad (33)$$

is the usual quantum number.

Our result describes the degeneracy of each level characterized by the value of the number n. For instance, $n = 0$ corresponds to the level ($l = 0$, $n' = 0$), $n = 1$ contains the three levels ($l = 1$, $n' = 0$), $n = 2$ contains the levels ($l = 0$, $n' = 1$) and ($l = 2$, $n' = 0$), etc. More generally if n is even, all even values of l are possible from 0 to n. If n is odd, all odd values of l are possible from 1 to n. Usually, the q-dimensional oscillator is described by the *invariance group* $SU(q)$ where each level n is associated with the "triangular" representation of symmetric tensors of rank n, the dimensionality of this representation describing the degree of degeneracy. One obtains in this way the reducibility of these representations of $SU(q)$ with respect to $SO(q)$.[6]

In this problem, the group of *invariance* $SO(q)$ *gives* rise to a *spatial degeneracy*. The *noninvariance group*[7] $SO(q) \times \overline{SU(1,1)}$ provides us with the complete spectrum, but the "accidental" degeneracy is not explained by an invariance group as it is in the case of the $SU(q)$ description.

III. THE RELATIVISTIC KEPLER PROBLEM

The radial Klein–Gordon equation for a Coulomb potential $V = -Ze^2/r$ can be put in the form[8]

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left(\frac{w}{4\rho} - \frac{1}{4} - \frac{l(l+1) - Z^2\alpha^2}{\rho^2}\right)R = 0, \quad (34)$$

where

$$\rho = (2/\hbar c)[m^2c^4 - E^2]^{\frac{1}{2}}r, \quad (35)$$

$$\alpha = e^2/\hbar c, \quad (36)$$

$$w = 4Z\alpha E/[m^2c^4 - E^2]^{\frac{1}{2}}. \quad (37)$$

---

[5] The authors are grateful to J.-J. Loeffel for his help in the understanding of this question. The property which is emphasized in the text is essentially due to the fact that one has for $K < -\frac{1}{4}$ two sets of square-integrable solutions of Eq. (1) instead of one when $K = 0$.

[6] The dimensionality of the space of homogeneous polynomials of degree l in the q-dimensional space is $\binom{l+q-1}{l}$. The Laplace operator transforms this space into the space of homogeneous polynomials of degree $l - 2$. The kernel of this transformation is the space of spherical harmonics of degree l, the dimensionality of which is then

$$N_l^q = \binom{l+q-1}{l} - \binom{l+q-3}{l-2} = (q + 2l - 2)\frac{(q+l-3)!}{l!(q-2)!}.$$

The reduction of the "triangular" representations of $SU(q)$ with respect to $SO(q)$ corresponds to the following relations

$$\sum_{\substack{l=0\\(\text{even})}}^{n}{}' N_l^q = \binom{n+q-1}{q}, \quad \sum_{\substack{l=1\\(\text{odd})}}^{n}{}' N_l^q = \binom{n+q-1}{q}.$$

[7] Clearly, we must use in our approach *reducible* representations of the group $SO(q) \times SU(I, I)$. We used the theory of spherical harmonics of the q-dimensional Euclidean space in order to obtain the reduction of the representation. In that sense, our method, although complete, is only *partially* group theoretical.

[8] The notation is that of L. J. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955) p. 338.

A new change of variable and function

$$\rho = x^2, \tag{38}$$

$$R = x^{-\frac{3}{2}}X, \tag{39}$$

leads to the Eq. (1) where K takes the value

$$K = 4l(l+1) - 4Z^2\alpha^2 + \tfrac{3}{4}. \tag{40}$$

The corresponding negative value[9] of σ is

$$\sigma = -\tfrac{1}{2} - [(l+\tfrac{1}{2})^2 - Z^2\alpha^2]^{\frac{1}{2}}. \tag{41}$$

The positiveness of the energy requires us to choose the representation $D^+(\sigma)$. Formulas (20), (35), and (39) provide us the energy spectrum

$$Z\alpha E/[m^2c^4 - E^2]^{\frac{1}{2}} = n' + \tfrac{1}{2} + [(l+\tfrac{1}{2})^2 - Z^2\alpha^2]^{\frac{1}{2}} \quad (n' = 0, 1, 2, \cdots), \tag{42}$$

a well-known result.

The same method can be used for the second-order Dirac equation

$$\{(p_\mu - eA_\mu)(p^\mu - eA^\mu) - (e\hbar/2c)\sigma^{\mu\nu}F_{\mu\nu} - m^2c^2\}\psi = 0, \tag{43}$$

which can be written (for the radial part) in the case of Coulomb potential

$$\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{2Z\alpha/\hbar c}{r} - \frac{m^2c^4 - E^2}{\hbar^2c^2} + \frac{Z^2\alpha^2 - l(l+1)/\hbar^2 - iZ\alpha\boldsymbol{\sigma}\cdot\hat{r}}{r^2}\right\}R = 0. \tag{44}$$

After the change of variables (35) and (37), this equation becomes

$$\left\{\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d}{d\rho}\right) + \left(\frac{w}{4\rho} - \frac{1}{4} + \frac{Z^2\alpha^2 - l(l+1)/\hbar^2 - iZ\alpha\boldsymbol{\sigma}\cdot\hat{r}}{\rho^2}\right)\right\}R = 0. \tag{45}$$

The last term has been diagonalized by Biedenharn[10] with the aid of the Martin–Glauber[11] operator

$$\Gamma = \frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{\hbar} + iZ\alpha\boldsymbol{\sigma}\cdot\hat{r} + 1 \tag{46}$$

[9] Z is supposed to be small ($Z < 68$).

[10] L. C. Biedenharn, Phys. Rev. **126**, 845 (1962).

[11] P. C. Martin and R. J. Glauber, Phys. Rev. **109**, 1307 (1958).

which satisfies the equation

$$\Gamma(\Gamma - 1) = \mathbf{L}^2/\hbar^2 + iZ\alpha\boldsymbol{\sigma}\cdot\hat{r} - Z^2\alpha^2. \tag{47}$$

The eigenvalues of $\Gamma(\Gamma-1)$ computed by Biedenharn are

$$[k^2 - Z^2\alpha^2]^{\frac{1}{2}}[(k^2 - Z^2\alpha^2)^{\frac{1}{2}} + \operatorname{sgn}(k)], \tag{48}$$

where $k = \pm 1, \pm 2, \pm 3, \cdots$. A simple calculation leads to the representations

$$D^+[-(k^2 - Z^2\alpha^2)^{\frac{1}{2}} - 1] \quad \text{for} \quad k > 0, \tag{49}$$

$$D^+[-(k^2 - Z^2\alpha^2)]^{\frac{1}{2}} \quad \text{for} \quad k < 0. \tag{50}$$

Consequently the energy spectrum is given by

$$\begin{aligned}\frac{Z\alpha E}{(m^2c^4 - E^2)^{\frac{1}{2}}} &= n' + 1 + (k^2 - Z^2\alpha^2)^{\frac{1}{2}} \quad (k > 0)\\ &= n' + (k^2 - Z^2\alpha^2)^{\frac{1}{2}} \quad (k < 0),\end{aligned} \tag{51}$$

in accordance with the usual formulas.

In this last example, we do not have, truly speaking, a dynamical group since the group which is involved, namely $SO(3) \times \overline{SU(1,1)}$ does not describe the degeneracy due to the spin. The spin variables increase the number of degrees of freedom and for this reason our group is not rich enough under this point of view. Some arguments could be found in favor of a group like $SO(5, 1)$ to classify the levels[12] in the Dirac case.

ACKNOWLEDGMENT

The authors are grateful to Dr. J.-J. Loeffel for helpful discussions and to Dr. E. H. Roffman for a critical reading of the manuscript.

[12] The q-dimensional nonrelativistic Kepler problem has, as a noninvariance group, the group $SO(q+1, 1)$. V. Fock, Z. Physik **98**, 145 (1935); V. Bargmann, *ibid.* **99**, 576 (1936); S. P. Alliluyev, Zh. Eksperim. i Teor. Fiz. **33**, 200 (1957) [English transl.: Soviet Phys.—JETP **6**, 260 (1957)]. The Lie algebra of this group can be realized with Poisson brackets. [For $q = 3$, see H. Bacry, Nuovo Cimento **41**, 222 (1966)]. A theorem in symplectic manifolds proves that the same Lie-Poisson algebra can be built for all q-degrees-of-freedom problems [see H. Bacry, H. Ruegg and J.-M. Souriau, Commun. Math. Phys. **3**, 323 (1966); H. Bacry, "Dynamical Groups and Relativity," lectures given in Boulder Colorado Summer Institute for Theoretical Physics, 1966 (to be published). See also direct proofs in N. Mukunda, "Dynamical Symmetries and Classical Mechanics, Realizations of Lie Algebras in Classical Mechanics," preprints, Syracuse University, Syracuse, New York, 1966; J. Rosen, "On Realizations of Lie Algebras and Symmetries in Classical and Quantum Mechanics," preprint; J.-M. Souriau, Compt. Rend. Acad. Sci. **263**, 1191(1966).]

If we use the definition of phase space given in H. Bacry, Commun. Math. Phys. **5**, 97 (1967), a particle with nonzero spin corresponds to a four-degrees-of-freedom problem. Consequently, one can *hope* to build explicitly the hydrogen-atom levels with the aid of a unitary representation of $SO(5, 1)$.

Discontinuities in Nonideal Magnetogasdynamics

HELMUT FRIEDEL*
Institute for Theoretical Physics, Innsbruck University, Innsbruck, Austria

(Received 14 November 1966)

An evolutionary condition is derived from generalized jump relations including dissipation terms and applied to the basic equations of magnetogasdynamics. Two contact discontinuities existing as solutions of these equations are found not to be evolutionary.

STARTING with Friedrichs,[1] hydromagnetic discontinuities were investigated in a most accurate way, within the framework of ideal magnetogasdynamics (MGD).[2–6] The ambiguity of such weak solutions is removed by a so-called evolutionary condition,[7,8] a first-order stability criterion. In the following, this condition is extended to nonideal, one-dimensional MGD.

The one-dimensional, unsteady system of equations, which is later on to be identified with the MGD basic equations, is assumed to be of the form

$$\mathbf{U}_t + (\mathbf{F} - s\mathbf{U})_x + \mathbf{G}_{xx} = 0. \tag{1}$$

The independent variables x, t refer to a coordinate system moving with the shock velocity s. The vectors $\mathbf{F}$ and $\mathbf{G}$ are continuous functions of the variables $\mathbf{U}$. The system (1) consists of a set of conservation laws which has been generalized by adding the dissipation terms $\mathbf{G}_{xx}$. For the shock relations connecting two states constant in space and time of the form

$$\begin{aligned} \bar{\mathbf{U}} = \mathbf{U}_1 &= \text{const} \quad (x < 0), \\ \mathbf{U}_2 &= \text{const} \quad (x > 0), \end{aligned} \tag{2}$$

we obtain from (1) by integration

$$[\bar{\mathbf{F}} - \bar{s}\bar{\mathbf{U}} + \mathbf{G}_x] = 0 \tag{3}$$

if $[Q] = Q_1 - Q_2$, $\bar{\mathbf{F}} = \mathbf{F}(\bar{\mathbf{U}})$, and $\bar{\mathbf{G}} = \mathbf{G}(\bar{\mathbf{U}})$. With respect to the subsequent identification of $\mathbf{G}$ (depending only on the magnetic field strength which must be continuous across a discontinuity in the case of finite electrical conductivity), (3) can only be satisfied if

$$[\bar{\mathbf{G}}] = 0. \tag{4}$$

The shock relations have, therefore, the general form

$$[\bar{\mathbf{F}} - \bar{s}\bar{\mathbf{U}}] = 0, \tag{5}$$

modified by the secondary condition (4). Discontinuities determined by these shock relations are required to satisfy an evolutionary condition. By virtue of the dissipation term, the condition we now derive is essentially different from that which holds for the nondissipative system.

A perturbed solution of (1) may be written in the form[9]

$$\mathbf{U} = \bar{\mathbf{U}} + \delta\mathbf{U} \tag{6}$$

with

$$s = \bar{s} + \delta s, \tag{7}$$

where the perturbations correspond to incoming and outgoing small-amplitude waves. $\mathbf{U}$ has to satisfy the boundary conditions

$$[\mathbf{F} - s\mathbf{U} + \mathbf{G}_x] = 0 \tag{8}$$

and the condition (4). Neglecting quadratic and higher-order terms in $\delta\mathbf{U}$, a Taylor expansion of $\mathbf{F}$ and $\mathbf{G}$ leads to the linearized form of (8):

$$\begin{aligned} &[\hat{A}\delta\mathbf{U} - \bar{\mathbf{U}}\delta s + R\delta\mathbf{U}_x] = 0, \\ &\qquad \hat{A} = A - sI,\ \delta\mathbf{U}_x = (\delta\mathbf{U})_x \end{aligned} \tag{9}$$

if the matrices A and R are defined as $A = \partial\mathbf{F}/\partial\mathbf{U}|_{\mathbf{U}=\bar{\mathbf{U}}}$ and $R = \partial\mathbf{G}/\partial\mathbf{U}|_{\mathbf{U}=\bar{\mathbf{U}}}$. In analogy to (3), (9) can only be fulfilled if

$$[R\delta\mathbf{U}] = 0. \tag{10}$$

Being a solution of the linearized equation (1), $\delta\mathbf{U}$ has the form

$$\delta\mathbf{U} = \sum_\alpha \delta a^{(\alpha)}\mathbf{r}^{(\alpha)}(w) \exp\left[iwt - ik^{(\alpha)}(w)x\right]. \tag{11}$$

Substituting (11) into the linearized equation (1), we obtain

$$(\hat{A}_j - Iw/k_j - iR_jk_j)\mathbf{r}_j = 0 \quad (j = 1, 2). \tag{12}$$

The index j refers to both sides of the discontinuity. Using (11) and (12), separating into incoming and outgoing waves ($\delta\mathbf{U}_{\text{in}}$, $\delta\mathbf{U}_{\text{out}}$), and choosing signs of

* Deceased.

[1] K. O. Friedrichs, Los Alamos Report No. 2105 (1954).

[2] A. I. Akhiezer, R. V. Polovin, and G. J. Lubarski, Zh. Eksperim. i Teor. Fiz. **35** (1958) [English transl.: Soviet Phys.—JETP **8**, 383 (1959)].

[3] A. I. Akhiezer, G. J. Lubarski, and R. V. Polovin, Proc. 2nd Intern. Conf. Peaceful Uses Atomic Energy, Geneva, 1958, **31**, 507 (1959).

[4] J. Bazer and O. Fleischmann, Phys. Fluids **2**, 366 (1959).

[5] J. Bazer and W. B. Ericson, Phys. Fluids **3**, 631 (1960).

[6] J. Bazer and W. B. Ericson, Astrophys. J. **129**, 758 (1960).

[7] P. D. Lax, Commun. Pure Appl. Math. **10**, 537 (1957).

[8] K. O. Friedrichs, *General Theory of High Speed Aerodynamics* (Oxford University Press, London, 1955), p. 33.

[9] We follow the treatment and notation of A. Jeffrey and T. Taniuti, *Non-linear Wave Propagation with Applications to Physics and Magnetohydrodynamics* (Academic Press Inc., New York, 1964).

$\delta a^{(\alpha)}_{\text{out,in}}$ appropriately, Eqs. (9) and (10) become

$$\sum_\alpha \delta a^{(\alpha)}_{\text{out}}\{(Iw/k^{(\alpha)})\mathbf{r}^{(\alpha)}\}_{\text{out}} + [\bar{\mathbf{U}}]\delta\sigma = \sum_\alpha \delta a^{(\alpha)}_{\text{in}}\{(Iw/k^{(\alpha)})\mathbf{r}^{(\alpha)}\}_{\text{in}}, \quad (13)$$

$$\sum_\alpha \delta a^{(\alpha)}_{\text{out}}(R\mathbf{r}^{(\alpha)})_{\text{out}} = \sum_\alpha \delta a^{(\alpha)}_{\text{in}}(R\mathbf{r}^{(\alpha)})_{\text{in}}, \quad (14)$$

where $\delta s = -\delta\sigma \exp(iwt)$. Equations (13) and (14) must be uniquely soluble with respect to $\delta a^{(\alpha)}_{\text{out}}$ and $\delta\sigma$ for given $\delta a^{(\alpha)}_{\text{in}}$ and arbitrary w, if the discontinuity is to be physically realizable. Since the wavenumbers $k^{(\alpha)}$ are complicated functions of w, as can be seen from the dispersion relation following from Eq. (12), the investigation of a long-wavelength perturbation, i.e., k vanishing in first order, proves to be useful. In first order, (12) goes over into the eigenvalue equation

$$(\hat{A}_j - Iw/k_j)\mathbf{r}_j = 0. \quad (15)$$

In this limit the phase velocities w/k_j exactly correspond to the characteristic velocities of the ideal equations (i.e., $\mathbf{G} = 0$).

Up to here the states $\mathbf{U}_j$ have been assumed to be constant. If now the $\mathbf{U}_j$ are allowed to be general continuous functions of x, it is easy to see that the boundary conditions are valid in the form just derived for $\mathbf{U}_j = \text{const}$, if the indices $j = 1, 2$ correspond only to the point $x = 0$ on both sides of the discontinuity. Also Eqs. (13) and (14) remain valid, since, as is shown later on, $R = \text{const}$.

The one-dimensional, unsteady MGD basic equations for constant electrical conductivity σ can be written in the form (1),[10] if

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v \\ \rho v^2/2 + \rho e + H^2/8\pi \\ \rho v_y \\ H_y \\ \rho v_z \\ H_z \end{bmatrix}, \quad (16)$$

$$\mathbf{F} = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p + H^2/8\pi \\ v_x(\rho v^2/2 + \rho e + p + H^2/4\pi) - (H_x/4\pi)\mathbf{v}\cdot\mathbf{H} \\ \rho v_y v_x - H_x H_y/4\pi \\ H_y v_x - v_y H_x \\ \rho v_z v_x - H_x H_z/4\pi \\ H_z v_x - v_z H_x \end{bmatrix}, \quad (17)$$

[10] H. Friedel, Scientific Report No. 30, Contract 61(052)-675, Innsbruck University (1966).

$$\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ (c^2/32\pi^2\sigma)H^2 \\ 0 \\ -(c^2/4\pi\sigma)H_y \\ 0 \\ -(c^2/4\pi\sigma)H_z \end{bmatrix}. \quad (18)$$

On both sides of the discontinuity, the magnetic permeability $\mu = 1$. $\mathbf{H} = (H_x, H_y, H_z)$ is the magnetic field strength, $\mathbf{v} = (v_x, v_y, v_z)$ the velocity, ρ the density, p the pressure, and e the internal energy density in Gaussian units (CGS units). With $\langle Q\rangle = \frac{1}{2}(Q_1 + Q_2)$, $m = \rho_1\hat{v}_{x1} = \rho_2\hat{v}_{x2}$, $\tau = 1/\rho$, and $\hat{v}_x = v_x - \bar{s}$, taking into account the secondary condition (4) [= the continuity condition for $\mathbf{H}$ as becomes patent from (18)], the jump relations (5) have the form

$$m[\tau] - [v_x] = 0, \quad (19)$$

$$m[v_x] + [p] = 0, \quad (20)$$

$$m[e + \langle p\rangle\tau] = 0, \quad (21)$$

$$m[v_y] = 0, \quad (22)$$

$$\langle H_y\rangle[v_x] - H_x[v_y] = 0, \quad (23)$$

$$m[v_z] = 0, \quad (24)$$

$$H_x[v_z] = 0. \quad (25)$$

If $m = 0$, Eqs. (19)–(25) determine two contact discontinuities:

$$\text{(a) } H_x \neq 0 \quad [p], [\mathbf{v}], [\mathbf{H}] = 0,$$

$$\text{(b) } H_x = 0 \quad [p], [v_x], [\mathbf{H}] = 0. \quad (26)$$

As is shown, both contact discontinuities cannot be evolutionary.

Case a. The solution of (15) shows that six different outgoing waves with the phase velocities $-V_{x1}$, $-c_{f1}$, $-c_{s1}$, V_{x2}, c_{f2}, c_{s2} exist,[11] where V_x is the Alfvén velocity based on H_x, c_f is the fast and c_s the slow magnetosonic velocity. Since $\mathbf{r}^{(\alpha)}$ and $[\bar{\mathbf{U}}]$ are linearly independent, a unique solution of (13) can be found.[11] From (18) we get

$$R = \begin{bmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && c^2/4\pi\sigma &&&& \\ &&& 0 &&& \\ &&&& -c^2/4\pi\sigma && \\ &&&&& 0 & \\ &&&&&& -c^2/4\pi\sigma \end{bmatrix}. \quad (27)$$

[11] A. Jeffrey and T. Taniuti, Ref. 9, Chap. 6.4.

Therefore (14) determines three further equations independent of (13), a fact that must give rise to a contradiction.

Case b. Without loss of generality the simplified case based on $\mathbf{v} = (v_x, 0, 0)$ and $\mathbf{H} = (0, H_y, 0)$ may be considered.[10,11] Two outgoing waves with the phase velocities c_{f1} $(= c_{s1})$ and $-c_{f2}$ $(= -c_{s2})$ lead to four independent equations (13), (14). Consequently, also in this case, the evolutionary condition cannot be fulfilled.

For $\mathbf{U}_j \neq$ constant contact $(m = 0)$ as well as general discontinuities, $(m \neq 0)$ may exist.[12] The general discontinuities are characterized by $[\mathbf{G}_x] \neq 0$. From what has been stated it is easy to see that both discontinuities cannot be evolutionary.

ACKNOWLEDGMENTS

The author thanks Professor F. Cap, head of the Institute, for encouragement and fruitful discussions.

This research has been sponsored in part by the United States Government under contract No. 61(052)–675.

[12] S. I. Pai, University of Maryland Technical Note BN-82 (1956).

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Effective Dielectric Tensor and Propagation Constant of Plane Waves in a Random Anisotropic Medium*

C. H. Liu
Department of Electrical Engineering, University of Illinois, Urbana, Illinois

(Received 13 January 1967)

Random anisotropic media are assumed to be characterized by dielectric tensors in which the components are random functions of position. A turbulent plasma in a static magnetic field is one example of such media. In this paper wave propagation in turbulent magnetoactive plasma is studied. The averages of electric field and dielectric displacement vectors over an ensemble of these media are found, assuming these average quantities are time-harmonic plane waves. The effective dielectric tensor is defined as the proportionality factor between the two average quantities. When this effective dielectric tensor is applied to the wave equation, a general dispersion relation for plane waves is derived. Expressions for propagation constants are obtained and some special cases are considered in detail. It is found that, because of random scattering, there are attenuations for both the ordinary and extraordinary waves for the average fields. The results reduce to those obtained by J. B. Keller and F. C. Karal when the anisotropy of the background media is removed

1. INTRODUCTION

RECENTLY, the problem of derivation of the effective dielectric constant, permeability, and conductivity of a random medium has been treated by several authors,[1–4] In these papers, the medium is assumed to consist of a uniform, isotropic background with random concentrations of small particles imbedded in it. But since in many cases the background medium is actually inhomogeneous or anisotropic—or even both—the problem of extending their results to such cases is of interest.

In this paper we limit ourselves to the study of the case of a special type of anisotropic medium corresponding to a plasma in a static magnetic field. The medium is characterized by a dielectric tensor $\boldsymbol{\epsilon}(\mathbf{x})$ which contains a random part. Keller and Karal's[2] method will be followed to derive an expression for the effective dielectric tensor operator as defined by them. In order to evaluate this tensor operator, the dyadic Green's function for this anisotropic background medium is derived explicitly in terms of a power series expansion. Using this Green's function, the dielectric tensor operator, when applied to a plane wave of wave vector $\mathbf{k}$, can be expressed as an ordinary tensor. General formulas are derived for the elements of this tensor. They reduce to the special case studied by Keller and Karal[1] when the background becomes isotropic. This effective dielectric tensor is used to predict the behavior of a plane wave by studying its

* This work was supported by National Aeronautics and Space Administration under Grant NSG 24.

[1] F. C. Karal and J. B. Keller, J. Math. Phys. 5, 537 (1964).
[2] J. B. Keller and F. C. Karal, J. Math. Phys. 7, 661 (1966).
[3] V. M. Finkel'berg, Zh. Techn. Fiz. 34, 509 (1964) [English transl.: Soviet Phys.—Tech. Phys. 9, 396, (1964)].
[4] Yu. A. Ryzhov, V. V. Tamoikin, and V. I. Tatarskil, Zh. Eksperim i Teor. Fiz. 48, 656 (1965) [English transl.: Soviet Phys.—JETP 21, 433 (1965)].

Therefore (14) determines three further equations independent of (13), a fact that must give rise to a contradiction.

Case b. Without loss of generality the simplified case based on $\mathbf{v} = (v_x, 0, 0)$ and $\mathbf{H} = (0, H_y, 0)$ may be considered.[10,11] Two outgoing waves with the phase velocities c_{f1} $(= c_{s1})$ and $-c_{f2}$ $(= -c_{s2})$ lead to four independent equations (13), (14). Consequently, also in this case, the evolutionary condition cannot be fulfilled.

For $\mathbf{U}_j \neq$ constant contact $(m = 0)$ as well as general discontinuities, $(m \neq 0)$ may exist.[12] The general discontinuities are characterized by $[\mathbf{G}_x] \neq 0$. From what has been stated it is easy to see that both discontinuities cannot be evolutionary.

ACKNOWLEDGMENTS

The author thanks Professor F. Cap, head of the Institute, for encouragement and fruitful discussions.

This research has been sponsored in part by the United States Government under contract No. 61(052)-675.

[12] S. I. Pai, University of Maryland Technical Note BN-82 (1956).

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Effective Dielectric Tensor and Propagation Constant of Plane Waves in a Random Anisotropic Medium*

C. H. Liu
Department of Electrical Engineering, University of Illinois, Urbana, Illinois

(Received 13 January 1967)

Random anisotropic media are assumed to be characterized by dielectric tensors in which the components are random functions of position. A turbulent plasma in a static magnetic field is one example of such media. In this paper wave propagation in turbulent magnetoactive plasma is studied. The averages of electric field and dielectric displacement vectors over an ensemble of these media are found, assuming these average quantities are time-harmonic plane waves. The effective dielectric tensor is defined as the proportionality factor between the two average quantities. When this effective dielectric tensor is applied to the wave equation, a general dispersion relation for plane waves is derived. Expressions for propagation constants are obtained and some special cases are considered in detail. It is found that, because of random scattering, there are attenuations for both the ordinary and extraordinary waves for the average fields. The results reduce to those obtained by J. B. Keller and F. C. Karal when the anisotropy of the background media is removed

1. INTRODUCTION

RECENTLY, the problem of derivation of the effective dielectric constant, permeability, and conductivity of a random medium has been treated by several authors,[1-4] In these papers, the medium is assumed to consist of a uniform, isotropic background with random concentrations of small particles imbedded in it. But since in many cases the background medium is actually inhomogeneous or anisotropic—or even both—the problem of extending their results to such cases is of interest.

In this paper we limit ourselves to the study of the case of a special type of anisotropic medium corresponding to a plasma in a static magnetic field. The medium is characterized by a dielectric tensor $\boldsymbol{\epsilon}(\mathbf{x})$ which contains a random part. Keller and Karal's[2] method will be followed to derive an expression for the effective dielectric tensor operator as defined by them. In order to evaluate this tensor operator, the dyadic Green's function for this anisotropic background medium is derived explicitly in terms of a power series expansion. Using this Green's function, the dielectric tensor operator, when applied to a plane wave of wave vector $\mathbf{k}$, can be expressed as an ordinary tensor. General formulas are derived for the elements of this tensor. They reduce to the special case studied by Keller and Karal[1] when the background becomes isotropic. This effective dielectric tensor is used to predict the behavior of a plane wave by studying its

* This work was supported by National Aeronautics and Space Administration under Grant NSG 24.

[1] F. C. Karal and J. B. Keller, J. Math. Phys. 5, 537 (1964).
[2] J. B. Keller and F. C. Karal, J. Math. Phys. 7, 661 (1966).
[3] V. M. Finkel'berg, Zh. Techn. Fiz. 34, 509 (1964) [English transl.: Soviet Phys.—Tech. Phys. 9, 396, (1964)].
[4] Yu. A. Ryzhov, V. V. Tamoikin, and V. I. Tatarskil, Zh. Eksperim i Teor. Fiz. 48, 656 (1965) [English transl.: Soviet Phys.—JETP 21, 433 (1965)].

dispersion relation. General expressions for propagation and attenuation constants for ordinary and extraordinary waves are derived. An example is given for the case of longitudinal propagation.

2. FORMULATION

Let us consider a turbulent plasma in a static magnetic field $\mathbf{B}_0$. The magnetic field is assumed to be in the z direction. For electromagnetic fields with harmonic time-variation $\exp(-i\omega t)$, the plasma can be characterized by the Cartesian dielectric tensor[5]

$$\boldsymbol{\epsilon}(\mathbf{x}) = \boldsymbol{\epsilon}_0 + \boldsymbol{\epsilon}_1(\mathbf{x}) = \begin{bmatrix} 1-X & -iXY & 0 \\ iXY & 1-X & 0 \\ 0 & 0 & 1-X \end{bmatrix} + \Delta X(\mathbf{x}) \begin{bmatrix} -1 & -iY & 0 \\ iY & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 0(Y^2), \quad (2.1)$$

where $X = \omega_p^2/\omega^2$, $Y = \omega_c/\omega$. ω_p and ω_c are the plasma frequency and cyclotron frequency of electrons, respectively. $\Delta X(\mathbf{x})$ represents the random concentration of irregularities in the plasma and is a random function of position. In the following we assume that the average of $\Delta X(\mathbf{x})$ over an ensemble of these media is zero, i.e., $\langle \Delta X(\mathbf{x}) \rangle = 0$. Also, in (2.1), $X < 1$ and $Y < 1$ are assumed.

For this dielectric tensor, the wave equation for the electric field is

$$L\mathbf{E} = (\nabla \times \nabla \times -k_0^2 \boldsymbol{\epsilon})\mathbf{E} = 0, \quad (2.2)$$

where k_0 is the free space wavenumber.

Following Keller and Karal,[1] Eq. (2.2) is written as

$$(L_0 - L_1)\mathbf{E} = 0, \quad (2.3)$$

where

$$L_0 = \nabla \times \nabla \times -k_0^2 \boldsymbol{\epsilon}_0, \quad (2.4)$$

$$L_1 = k_0^2 \boldsymbol{\epsilon}_1. \quad (2.5)$$

The solution of Eq. (2.3) can be expressed as a series given by

$$\mathbf{E} = \mathbf{E}_0 + L_0^{-1}L_1\mathbf{E}_0 + L_0^{-1}L_1L_0^{-1}L_1\mathbf{E}_0 + 0((\Delta X)^3), \quad (2.6)$$

where $\mathbf{E}_0$ is the solution of (2.3) when $L_1 = 0$ and L_0^{-1} is the inverse operator of L_0. From the definition of dielectric tensor, the displacement vector $\mathbf{D}$ is obtained by multiplying (2.6) by $\boldsymbol{\epsilon}$.

$$\begin{aligned} \mathbf{D} &= \boldsymbol{\epsilon}\mathbf{E} \\ &= \boldsymbol{\epsilon}_0\mathbf{E}_0 + (\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_0 L_0^{-1}L_1)\mathbf{E}_0 \\ &\quad + (\boldsymbol{\epsilon}_1 L_0^{-1}L_1 + \boldsymbol{\epsilon}_0 L_0^{-1}L_1L_0^{-1}L_1)\mathbf{E}_0 + 0((\Delta X)^3) \quad (2.7) \end{aligned}$$

which holds true for each sample of the ensemble. Taking the ensemble average of (2.7) and using the assumption $\langle \Delta X \rangle = 0$, we obtain

$$\langle \mathbf{D} \rangle = \boldsymbol{\epsilon}_0\mathbf{E}_0 + (\langle \boldsymbol{\epsilon}_1 L_0^{-1}L_1 \rangle + \boldsymbol{\epsilon}_0 \langle L_0^{-1}L_1L_0^{-1}L_1 \rangle)\mathbf{E}_0 + 0((\Delta X)^3). \quad (2.8)$$

Similarly, the average of Eq. (2.6) becomes

$$\langle \mathbf{E} \rangle = \mathbf{E}_0 + \langle L_0^{-1}L_1L_0^{-1}L_1 \rangle \mathbf{E}_0 + 0((\Delta X)^3). \quad (2.9)$$

This equation is then solved by iteration to yield

$$\mathbf{E}_0 = \langle \mathbf{E} \rangle - \langle L_0^{-1}L_1L_0^{-1}L_1 \rangle \langle \mathbf{E} \rangle + 0((\Delta X)^3). \quad (2.10)$$

Substituting Eq. (2.10) into (2.8), we obtain

$$\langle \mathbf{D} \rangle = \boldsymbol{\epsilon}_0 \langle \mathbf{E} \rangle + \langle \boldsymbol{\epsilon}_1 L_0^{-1}L_1 \rangle \langle \mathbf{E} \rangle + 0((\Delta X)^3). \quad (2.11)$$

If we define the effective dielectric tensor by

$$\langle \mathbf{D} \rangle = \boldsymbol{\epsilon}_{\text{eff}} \langle \mathbf{E} \rangle \quad (2.12)$$

then, from Eq. (2.11), we have

$$\boldsymbol{\epsilon}_{\text{eff}} = \boldsymbol{\epsilon}_0 + \langle \boldsymbol{\epsilon}_1 L_0^{-1}L_1 \rangle + 0((\Delta X)^3). \quad (2.13)$$

The above was derived by Keller and Karal.[2] From this we notice that the effective dielectric tensor is, in general, in the form of an operator tensor. In the following we concentrate on the evaluation of the operator tensor $\langle \boldsymbol{\epsilon}_1 L_0^{-1}L_1 \rangle$.

3. INVERSE OPERATOR L_0^{-1}

In terms of Green's function, the inverse operator L_0^{-1} operating on any vector function $\mathbf{F}(\mathbf{x})$ can be represented by

$$L_0^{-1}F(\mathbf{x}) = \int \boldsymbol{\Gamma}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{F}(\mathbf{x}')\, d\mathbf{x}', \quad (3.1)$$

where $\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{x}')$ is the dyadic Green's function satisfying the equation

$$L_0\boldsymbol{\Gamma} = -\mathsf{I}\delta(\mathbf{x} - \mathbf{x}'), \quad (3.2)$$

where I is the unit dyadic. A dot represents scalar product. The Fourier transform of (3.2) yields

$$[(k_1^2 - p^2)\mathsf{I} + \mathbf{pp} + k_0^2\boldsymbol{\eta}] \cdot \tilde{\boldsymbol{\Gamma}}(\mathbf{p}) = \mathsf{I}, \quad (3.3)$$

where

$$k_1^2 = k_0^2(1 - X), \quad (3.4)$$

$$\boldsymbol{\eta} = iXY \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (3.5)$$

and

$$\tilde{\boldsymbol{\Gamma}}(\mathbf{p}) = \int \boldsymbol{\Gamma}(\mathbf{x}, \mathbf{x}') e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\, d\mathbf{x}. \quad (3.6)$$

Equation (2.1) is used in deriving (3.3). For high frequency, $Y \ll 1$, Eq. (3.3) can be solved in a series

[5] C. H. Papas, *Theory of Electromagnetic Wave Propagation* (McGraw-Hill Book Company, Inc., New York., 1965), p. 189.

of ascending powers of Y. Let

$$\tilde{\Gamma}_{ij} = \tilde{\Gamma}^0_{ij} + \tilde{\Gamma}^1_{ij} + 0(Y^2), \tag{3.7}$$

where

$$|\tilde{\Gamma}^1_{ij}|/|\tilde{\Gamma}^0_{ij}| = 0(Y) \ll 1. \tag{3.8}$$

Equation (3.3) can be decomposed into

*Zero*th order:

$$[(k_1^2 - p^2)\delta_{ik} + p_i p_k]\tilde{\Gamma}^0_{kj} = \delta_{ij}, \tag{3.9}$$

First order:

$$[(k_1^2 - p^2)\delta_{ik} + p_i p_k]\tilde{\Gamma}^1_{kj} = -k_0^2\eta_{ik}\tilde{\Gamma}^0_{kj}. \tag{3.10}$$

The solutions for these two algebraic equations are respectively

$$\tilde{\Gamma}^0_{ij} = (-\delta_{ij} + p_i p_j/k_1^2)/(p^2 - k_1^2), \tag{3.11}$$

$$\tilde{\Gamma}^1_{ij} = [-k_0^2\eta_{ij} + (k_0^2/k_1^2)\eta_{ik}p_k p_j + (k_0^2/k_1^2)p_i p_k \eta_{kj} - (k_0^2/k_1^4)p_i p_k \eta_{kl} p_l p_j]/(p^2 - k_1^2)^2. \tag{3.12}$$

When the values of the components of the matrix $\boldsymbol{\eta}$ are substituted into (3.12), we have

$$\tilde{\Gamma}^1_{12} = (jXYk_0^2/k_1^2)[k_1^2 - (p_1^2 + p_2^2)]/(p^2 - k_1^2)^2 = -\tilde{\Gamma}^1_{21}, \tag{3.13}$$

$$\tilde{\Gamma}^1_{13} = (-jXYk_0/k_1)p_2 p_3/(p^2 - k_1)^2 = -\tilde{\Gamma}^1_{31}, \tag{3.14}$$

$$\tilde{\Gamma}^1_{23} = (jXYk_0^2/k_1^2)p_3 p_1/(p^2 - k_1^2)^2 = -\tilde{\Gamma}^1_{32}, \tag{3.15}$$

$$\tilde{\Gamma}^1_{11} = \tilde{\Gamma}^1_{22} = \tilde{\Gamma}^1_{33} = 0. \tag{3.16}$$

The inverse Fourier transforms of these expressions can be obtained with the aid of the mean value theorems.[2] Details are given in Appendix A. To the first order in Y, the dyadic Green's function is given by

$$\Gamma_{ij}(\mathbf{x}, \mathbf{x}') = \Gamma^0_{ij}(\mathbf{x}, \mathbf{x}') + \Gamma^1_{ij}(\mathbf{x}, \mathbf{x}'), \tag{3.17}$$

where

$$\Gamma^0_{ij}(\mathbf{x}, \mathbf{x}') = G_1(r)\delta_{ij} + G_2(r)r_i r_j/r^2, \tag{3.18}$$

$$\Gamma^1_{12}(\mathbf{x}, \mathbf{x}') = [ik_1 r - 2 - (ik_1 r - 1)(r_1^2 + r_2^2)/r^2]G_3(r) = -\Gamma^1_{21}(\mathbf{x}, \mathbf{x}'), \tag{3.19}$$

$$\Gamma^1_{13}(\mathbf{x}, \mathbf{x}') = -[(ik_1 r - 1)(r_2^2 + r_3^2)/r^2]G_3(r) = -\Gamma^1_{31}(\mathbf{x}, \mathbf{x}'), \tag{3.20}$$

$$\Gamma^1_{23}(\mathbf{x}, \mathbf{x}') = [(ik_1 r - 1)(r_1^2 + r_3^2)/r^2]G_3(r) = -\Gamma^1_{32}(\mathbf{x}, \mathbf{x}'), \tag{3.21}$$

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^1_{33} = 0. \tag{3.22}$$

For convenience we have adopted the following notations:

$$\mathbf{r} = \mathbf{x} - \mathbf{x}', \quad r = |\mathbf{r}|, \tag{3.23}$$

$$G_1(r) = (-1 + ik_1 r + k_1^2 r^2)e^{ik_1 r}/4\pi k_1^2 r^3 - \delta(r)/12\pi k_1^2 r^2, \tag{3.24}$$

$$G_2(r) = (3 - 3ik_1 r - k_1^2 r^2)e^{ik_1 r}/4\pi k_1^2 r^3, \tag{3.25}$$

$$G_3(r) = iXYk_0^2 e^{ik_1 r}/8\pi k_1^2 r. \tag{3.26}$$

One may notice here that this Green's function is valid in a region where the inequality $|\Gamma^1_{ij}| \ll |\Gamma^0_{ij}|$ is satisfied. This corresponds to a region not far away from the source, so that the Faraday rotation effect of the field is not important. In the evaluation of the effective dielectric tensor, the most important contribution to the scattered field from the irregularities is the near-field contribution. Therefore, no appreciable error is introduced if this Green's function is used. The same procedure can be followed to get higher-order terms for the Green's function.

4. EFFECTIVE DIELECTRIC TENSOR

To evaluate the effective dielectric tensor, we start with the evaluation of $\langle \boldsymbol{\epsilon}_1 L_0^{-1} L_1 \rangle$, which is an operator tensor. When it is applied to a plane wave $\mathbf{A}\exp(i\mathbf{k}\cdot\mathbf{x})$, it becomes, with the help of (3.1),

$$\begin{aligned}
\langle \boldsymbol{\epsilon}_1 L_0^{-1} L_1 \rangle \cdot \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{x}'}
&= k_0^2 \langle \boldsymbol{\epsilon}_1(\mathbf{x}) \cdot \int \boldsymbol{\Gamma}(\mathbf{x}, \mathbf{x}') \cdot \boldsymbol{\epsilon}_1(\mathbf{x}') \cdot \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \rangle \\
&= k_0^2 \langle \boldsymbol{\epsilon}_1(\mathbf{x}) \cdot \int \boldsymbol{\Gamma}^0(\mathbf{x}, \mathbf{x}') \cdot \boldsymbol{\epsilon}_1(\mathbf{x}') \cdot \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \rangle \\
&\quad + k_0^2 \langle \boldsymbol{\epsilon}_1(\mathbf{x}) \cdot \int \boldsymbol{\Gamma}^1(\mathbf{x}, \mathbf{x}') \cdot \boldsymbol{\epsilon}_1(\mathbf{x}') \cdot \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \rangle \\
&\quad + 0(Y^2(\Delta X)^3).
\end{aligned} \tag{4.1}$$

In component form, Eq. (4.1) can be written:

$$\begin{aligned}
\langle \boldsymbol{\epsilon}_1 L_0^{-1} L_1 \rangle_{ij} A_j e^{i\mathbf{k}\cdot\mathbf{x}'}
&= k_0^2 \langle \epsilon_{lin}(\mathbf{x}) \int \Gamma^0_{nm}(\mathbf{x}, \mathbf{x}')\epsilon_{lmj}(\mathbf{x}') A_j e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \rangle \\
&\quad + k_0^2 \langle \epsilon_{lin}(\mathbf{x}) \int \Gamma^1_{nm}(\mathbf{x}, \mathbf{x}')\epsilon_{lmj}(\mathbf{x}') A_j e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \rangle \\
&= k_0^2 C_{inmj}\left[\int \Gamma^0_{nm}(\mathbf{x}, \mathbf{x}')\rho(r) A_j e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' + \int \Gamma^1_{nm}(\mathbf{x}, \mathbf{x}')\rho(r) A_j e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \right],
\end{aligned} \tag{4.2}$$

where

$$C_{inmj}\rho(r) = \langle \epsilon_{lin}(\mathbf{x})\epsilon_{lmj}(\mathbf{x}') \rangle. \tag{4.3}$$

The random part of the dielectric tensor $\boldsymbol{\epsilon}_1(\mathbf{x})$, as given by (2.1), is due entirely to random fluctuations in plasma density whose correlation is

$$\rho(r) = \langle \Delta X(\mathbf{x})\Delta X(\mathbf{x}') \rangle / \langle (\Delta X)^2 \rangle. \tag{4.4}$$

For simplicity we have assumed that irregularities in the density are homogeneous and isotropic, so that the correlation function depends only on r.

The tensor product $C_{inmj}R_{nm}$ for any tensor R_{nm} is given in Appendix B. Applying this relation on (4.2), using the mean value theorem developed by

Keller and Karal,[1] we can write

$$\langle \boldsymbol{\epsilon}_1 L_0^{-1} L_1 \rangle_{ij} A_j e^{i\mathbf{k}\cdot\mathbf{x}'} = (S_{ij} + T_{ij}) A_j e^{i\mathbf{k}\cdot\mathbf{x}} \quad (4.5)$$

where the right-hand side, S_{ij} and T_{ij}, are only tensors and no longer operators. Elements of the S matrix are related to the zeroth-order Green's dyadic $\mathbf{\Gamma}^0$, while elements of the T matrix are related to the first-order Green's dyadic $\mathbf{\Gamma}^{(1)}$. They are given to the order of $Y\langle(\Delta X)^2\rangle$ by:

$$\begin{aligned}
S_{11} &= \langle(\Delta X)^2\rangle[D + (k_x^2/k^2)M],\\
S_{22} &= \langle(\Delta X)^2\rangle[D + (k_y^2/k^2)M],\\
S_{33} &= \langle(\Delta X)^2\rangle[D + (k_z^2/k^2)M],\\
S_{12} &= \langle(\Delta X)^2\rangle[(k_x k_y/k^2)M + i2YD\\
&\qquad + iY(k_x^2 + k_y^2)M/k^2],\\
S_{13} &= \langle(\Delta X)^2\rangle[M(k_x k_z + iYk_y k_z)/k^2], \qquad (4.6)\\
S_{21} &= \langle(\Delta X)^2\rangle[(k_x k_z/k^2)M - i2YD\\
&\qquad - iY(k_x^2 + k_y^2)M/k^2],\\
S_{23} &= \langle(\Delta X)^2\rangle[M(k_y k_z - iYk_x k_z)/k^2],\\
S_{31} &= \langle(\Delta X)^2\rangle[M(k_x k_z - iYk_y k_z)/k^2],\\
S_{32} &= \langle(\Delta X)^2\rangle[M(k_y k_z + iYk_x k_z)/k^2];
\end{aligned}$$

$$\begin{aligned}
T_{11} &= T_{22} = T_{33} = 0,\\
T_{12} &= \langle(\Delta X)^2\rangle[F - H_x - H_y] = -T_{21},\\
T_{13} &= -\langle(\Delta X)^2\rangle[H_y + H_z] = -T_{31},\\
T_{23} &= \langle(\Delta X)^2\rangle[H_x + H_z] = -T_{32},
\end{aligned} \quad (4.7)$$

where $\mathbf{k}$ is the wave vector for the plane wave, and

$$D = k_0^2 \int_0^\infty \left[G_1(r)\rho(r)f(r) - G_2(r)\rho(r)r^{-2}k^{-1}\frac{\partial f}{\partial k}\right] dr, \quad (4.8)$$

$$M = -k_0^2 \int_0^\infty G_2(r)\rho(r)r^{-2}\left(\frac{\partial^2 f}{\partial k^2} - k^{-1}\frac{\partial f}{\partial k}\right) dr, \quad (4.9)$$

$$F = k_0^2 \int_0^\infty (ik_1 r - 2)G_3(r)\rho(r)f(r)\, dr, \quad (4.10)$$

$$H_\alpha = -k_0^2 \int_0^\infty (ik_1 r - 1)G_3(r)\rho(r)r^{-2} \times \left[\frac{k_\alpha^2}{k^2}\frac{\partial^2 f}{\partial k^2} + \frac{1}{k}\frac{\partial f}{\partial k}\left(1 - \frac{k_\alpha^2}{k^2}\right)\right] dr, \quad \alpha = x, y, z \quad (4.11)$$

$$f(r, k) = \frac{4\pi r}{k}\sin kr. \quad (4.12)$$

$G_1(r)$, $G_2(r)$, and $G_3(r)$ are given by (3.24), (3.25), and (3.26), respectively.

To prove Eqs. (4.6) and (4.7), we consider the following component of Eq. (4.2):

$$\begin{aligned}
\langle \boldsymbol{\epsilon}_1 L_0^{-1} L_1\rangle_{12} A_2 e^{i\mathbf{k}\cdot\mathbf{x}'}
&= \langle(\Delta X)^2\rangle k_0^2 \int [\Gamma_{12} + iY(\Gamma_{11} + \Gamma_{22})]\rho(r) A_2 e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}'\\
&= \langle(\Delta X)^2\rangle k_0^2 \int \left\{ \frac{[r_1 r_2 + iY(r_1^2 + r_2^2)]G_2(r)}{r^2} + i2YG_1(r) \right.\\
&\qquad \left. + \left[\frac{ik_1 r - 2 - (ik_1 r - 1)(r_1^2 + r_2^2)}{r^2}\right] G_3(r)\right\}\\
&\qquad \times \rho(r) A_2 e^{i\mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}'. \qquad (4.13)
\end{aligned}$$

Changing the variable of integration by the relation $x' = s - r$, writing $dr = dr\, dS$ where dS is the area element on a sphere of radius r centered at x, and applying the mean value theorem, we obtain for (4.13)

$$\begin{aligned}
\langle \boldsymbol{\epsilon}_1 L_0^{-1} L_1\rangle_{12} A_2 e^{i\mathbf{k}\cdot\mathbf{x}'}
&= \langle(\Delta X)^2\rangle\{(k_x k_y/k^2)M + iY[2D + (k_x^2 + k_y^2)M/k^2]\\
&\qquad + [F - H_x - H_y]\} A_2 e^{i\mathbf{k}\cdot\mathbf{x}}\\
&= (S_{12} + T_{12}) A_2 e^{i\mathbf{k}\cdot\mathbf{x}}. \qquad (4.14)
\end{aligned}$$

Therefore, we have obtained the components S_{12} and T_{12}. The other terms of the tensors can be calculated in a similar manner.

Summarizing the above results, we have the effective dielectric tensor for a plane wave with wavenumber $\mathbf{k}$:

$$\epsilon_{\mathrm{eff}ij} = \epsilon_{0ij} + S_{ij} + T_{ij}. \quad (4.15)$$

We note in (4.15) that the effective dielectric tensor contains a Hermitian part ϵ_{0ij} and a non-Hermitian part $S_{ij} + T_{ij}$. The former is the dielectric tensor of a plasma in an external magnetic field $\mathbf{B}_0$ and satisfies Onsager's relation $\epsilon_{0ij}(\mathbf{B}_0) = \epsilon_{0ij}(-\mathbf{B}_0)$.[5] The latter is the contribution to the anisotropy from random scattering of waves in the medium and, in general, does not satisfy Onsager's relation. From the non-Hermitian property of the effective dielectric tensor $\boldsymbol{\epsilon}_{\mathrm{eff}}$ it is evident that attenuations of the average fields will occur in the medium and they are caused by the random scattering of waves. Attenuation constants will be derived in the next section.

Also we note that the elements of the effective dielectric tensor depend on the wave vector $\mathbf{k}$. For the case $Y = 0$, so that the background medium is isotropic, we have $\mathbf{\Gamma}^1 = 0$, and the above results reduce to those for a random, isotropic medium treated in Ref. 2.

5. PROPAGATION CONSTANT

Once the effective dielectric tensor for the average field in this medium is known, the general dispersion relation for a plane wave with wave vector $\mathbf{k}$ can be derived. From Eqs. (2.2) and (2.6), it is easy to show[1] that the average field satisfies

$$L_0\langle\mathbf{E}\rangle + L_1L_0^{-1}L_1\langle\mathbf{E}\rangle = 0. \qquad (5.1)$$

For a plane wave of the form

$$\langle\mathbf{E}\rangle = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}},$$

Eq. (5.1) reduces to

$$\mathsf{W}(\mathbf{k})\cdot\mathbf{E}_0 = 0, \qquad (5.2)$$

where

$$\mathsf{W}(\mathbf{k}) = \mathbf{kk} - k^2\mathsf{I} + k_0^2\boldsymbol{\epsilon}_{\text{eff}}. \qquad (5.3)$$

The dispersion relation for this plane wave is then given by the equation

$$\det \overline{\mathsf{W}}\,(\mathbf{k}) = 0, \qquad (5.4)$$

where det represents determinant. Substituting (4.13) and (5.3) into (5.4), we obtain the following:

$$\begin{aligned}(k/k_0)^4[&a_{11}\sin^2\theta\cos^2\phi + a_{22}\sin^2\theta\sin^2\phi + a_{33}\cos^2\theta\\ &+ (a_{13}+a_{31})\sin\theta\cos\theta\cos\phi + (a_{32}+a_{23})\\ &\times\sin\theta\cos\theta\sin\phi + (a_{12}+a_{21})\\ &\times\sin^2\theta\cos\phi\sin\phi] - (k/k_0)^2\{[a_{22}(a_{13}+a_{31})\\ &- a_{12}a_{23} - a_{21}a_{32}]\sin\theta\cos\theta\sin\phi\\ &+ a_{33}(a_{12}+a_{21})\sin^2\theta\cos\phi\sin\phi\\ &- a_{22}a_{33}\sin^2\theta\cos^2\phi - a_{11}a_{33}\sin^2\theta\sin^2\phi\\ &- (a_{11}a_{22}-a_{12}a_{21})\cos^2\theta + a_{11}a_{33} + a_{22}a_{33}\\ &+ a_{11}a_{22} - a_{12}a_{21}\} + (a_{11}a_{22}-a_{12}a_{21})a_{33} = 0,\end{aligned} \qquad (5.5)$$

where we have defined

$$\begin{aligned}k_x &= k\sin\theta\cos\phi,\\ k_y &= k\sin\theta\sin\phi,\\ k_z &= k\cos\theta,\\ a_{ij} &= \epsilon_{\text{eff }ij}.\end{aligned} \qquad (5.6)$$

For the case when there is no randomness so that $\Delta X = 0$, Eq. (5.5) reduces to the well-known dispersion relation for a magnetoionic medium. The wavenumbers are then given by[5]

$$\left(\frac{k_{\mathrm{I}}}{k_0}\right)^2 = 1 - \frac{X}{1 - Y^2\sin^2\theta/2(1-X) + [Y^4\sin^4\theta/4(1-X)^2 + Y^2\cos^2\theta]^{\frac{1}{2}}},$$

$$\left(\frac{k_{\mathrm{II}}}{k_0}\right)^2 = 1 - \frac{X}{1 - Y^2\sin^2\theta/2(1-x) - [Y^4\sin^4\theta/4(1-X)^2 + Y^2\cos^2\theta]^{\frac{1}{2}}}, \qquad (5.7)$$

where k_{I} and k_{II} are the propagation constants for ordinary and extraordinary waves, respectively.

For high frequencies, and quasilongitudinal waves (waves propagating outside a small region about $\theta \approx 90°$), Eq. (5.7) yields

$$\begin{aligned}(k_{\mathrm{I}}/k_0)_0^2 &= 1 - X + XY\cos\theta,\\ (k_{\mathrm{II}}/k_0)_0^2 &= 1 - X - XY\cos\theta.\end{aligned} \qquad (5.8)$$

Both modes have electric fields circularly polarized in a plane perpendicular to $\mathbf{k}$. The corrections to k_{I} and k_{II}, due to the small randomness ΔX, can be obtained by solving Eq. (5.5) using perturbation method. We obtain, to the order of $Y\langle(\Delta X)^2\rangle$,

$$\begin{aligned}(k_{\mathrm{I}}/k_0)^2 &= n_{\mathrm{I}} + n_{\mathrm{I}}^1,\\ (k_{\mathrm{II}}/k_0)^2 &= n_{\mathrm{II}} + n_{\mathrm{II}}^1,\end{aligned} \qquad (5.9)$$

where

$$n_{\mathrm{I}} = 1 - X + XY\cos\theta, \qquad (5.10)$$

$$n_{\mathrm{I}}^1 = \frac{n_{\mathrm{I}}^2\,\delta_1 - n_{\mathrm{I}}\cdot\delta_2 + \delta_3}{2(1-X)XY\cos\theta}, \qquad (5.11)$$

$$n_{\mathrm{II}} = 1 - X - XY\cos\theta, \qquad (5.12)$$

$$n_{\mathrm{II}}^1 = -\frac{n_{\mathrm{II}}^2\,\delta_1 - n_{\mathrm{II}}\,\delta_2 + \delta_3}{2(1-X)XY\cos\theta}. \qquad (5.13)$$

In the Appendix δ_1, δ_2, δ_3 are given; they correspond to the correction terms in the coefficients of Eq. (5.5). They are functions of the magnitude as well as the direction of the vector $\mathbf{k}$. In general, Eq. (5.9) must be solved for k in which these functions are substituted.

For the special case of longitudinal propagation, so that $\theta = 0$, the expressions δ_1, δ_2, and δ_3 reduce to

$$\delta_1 = Z_{33}, \qquad (5.14)$$

$$\delta_2 = (1-X)(Z_{11} + Z_{22} + 2Z_{33}), \qquad (5.15)$$

$$\delta_3 = (1-X)^2(Z_{11} + Z_{22} + Z_{33}) + i(1-X)XY(Z_{21} - Z_{12}), \qquad (5.16)$$

where we have written

$$Z_{ij} = S_{ij} + T_{ij} \qquad (5.17)$$

for simplicity. For this case, Eqs. (5.11) and (5.13) become

$$n_{\mathrm{I}}^1 = \tfrac{1}{2}[Z_{11} + Z_{22} - i(Z_{21} - Z_{12})], \qquad (5.18)$$

$$n_{\mathrm{II}}^1 = \tfrac{1}{2}[Z_{11} + Z_{22} + i(Z_{21} - Z_{12})]. \qquad (5.19)$$

Substituting (5.18) and (5.19) into (5.9), we obtain the equations for the propagation constants for ordinary and extraordinary waves, respectively, up to the order $Y\langle(\Delta X)^2\rangle$:

$$(k_{\mathrm{I}}/k_0)^2 = (1-X) + XY + \langle(\Delta X)^2\rangle \times [D + i(2iYD + F - H_x - H_y)]; \qquad (5.20)$$

$$(k_{\mathrm{II}}/k_0)^2 = (1-X) - XY + \langle(\Delta X)^2\rangle \times [D - i(2iYD + F - H_x - H_y)]. \qquad (5.21)$$

These two equations reduce to a single one when $Y = 0$, corresponding to the equation derived by

Keller and Karal[1] for the case in which the electric field is perpendicular to the direction of propagation. When $Y \neq 0$, because of the anisotropic property of the background medium, there are two modes of propagation. The propagation constants for each mode is affected by the randomness of the medium in a different way. In general, to the order of $Y\langle(\Delta X)^2\rangle$, the solution of Eq. (5.20) can be obtained simply by substituting $k_0(n_{\rm I})^{\frac{1}{2}}$ for k in the evaluation of D, F, and H's on the right-hand side of (5.20). Similarly, the substitution of $k_0(n_{\rm II})^{\frac{1}{2}}$ for k on the right-hand side of Eq. (5.21) gives the solution for $k_{\rm II}$ to this order.

The attenuation constants for these two modes can then be written as

$$\alpha_{\rm I} = I_m(k_{\rm I}) = \frac{k_0\langle(\Delta X)^2\rangle}{2(1-X)^{\frac{1}{2}}} I_m D + R_e[2iYD + (F - H_x - H_y)], \quad (5.22)$$

$$\alpha_{\rm II} = I_m(k_{\rm II}) = \frac{k_0\langle(\Delta X)^2\rangle}{2(1-X)^{\frac{1}{2}}} I_m D - R_e[2iYD + (F - H_x - H_y)], \quad (5.23)$$

both to the order of $Y\langle(\Delta X)^2\rangle$.

As an example, let us consider the special autocorrelation function given by

$$\rho(r) = \exp(-a^{-1}r). \quad (5.24)$$

Substituting (5.24) into (4.8)–(4.11), we obtain

$$D = \frac{k_0^2}{2k_1^2}\left[\frac{1+ik_1a}{k^2a^2} + \frac{2(k_1a)^2}{(ka)^2+(1-ik_1a)^2} - \frac{1}{ka}\left(1+\frac{1+k_1^2a^2}{k^2a^2}\right)\cot^{-1}\left(\frac{1-ik_1a}{ka}\right)\right], \quad (5.25)$$

$$F = \frac{-iXYk_0^4a^2}{k_1^2}\,\frac{a^2(k^2-2k_1^2)-3iak_1}{[a^2(k_1^2-k^2)+2iak]^2}, \quad (5.26)$$

$$H_x + H_y = -\frac{iXYk_0^4}{k_1^2k^2} \times\left[1 - \frac{1-iak_1}{ak}\cot^{-1}\left(\frac{1-ik_1a}{ka}\right)\right]. \quad (5.27)$$

By setting $k = k_0(n_{\rm I})^{\frac{1}{2}}$ and $k = k_0(n_{\rm II})^{\frac{1}{2}}$ in Eqs. (5.25)–(5.27) and substituting them into (5.22) and (5.23), respectively, we obtain the attenuation constants $\alpha_{\rm I}$ and $\alpha_{\rm II}$. For $\langle(\Delta X)^2\rangle\langle k_1a\rangle \ll 1$, Eqs. (5.25), (5.26), and (5.27) can be simplified, and Eqs. (5.20) and (5.21) yield

$$k_{\rm I} = k_0(1-X)^{\frac{1}{2}}\left[1 + \frac{XY}{2(1-X)} + i\frac{\langle(\Delta X)^2\rangle}{4(1-X)^2} \times\left(\frac{1-2Y}{k_1a} + \frac{3}{2}\frac{XYak_1}{1-X}\right)\right] + 0(Y^2(\Delta X)^3), \quad (5.28)$$

$$k_{\rm II} = k_0(1-X)^{\frac{1}{2}}\left[1 - \frac{XY}{2(1-X)} + i\frac{\langle(\Delta X)^2\rangle}{4(1-X)^2} \times\left(\frac{1+2Y}{k_1a} - \frac{3}{2}\frac{XYak_1}{1-X}\right)\right] + 0(Y^2(\Delta X)^3). \quad (5.29)$$

We see that for cases where the correlation length of the medium is small $k_1a \ll 1$, the propagation constants to the order of $Y\langle(\Delta X)^2\rangle$ are the same as those for a nonrandom medium. The attenuation constants are of the order $\langle(\Delta X)^2\rangle$ and are different for ordinary and for extraordinary waves. For cases such that $k_1a \gg 1$, we can no longer substitute $k_0(n_{\rm I})^{\frac{1}{2}}$ or $k_0(n_{\rm II})^{\frac{1}{2}}$ in Eqs. (5.25)–(5.27) because some of the terms become very large. Therefore, Eqs. (5.20) and (5.21) must be solved more accurately. In principle, a perturbation method can be used in which the solution given by Keller and Karal[1] will be taken as the zeroth-order solution for $Y = 0$. Higher-order terms can be obtained for $Y \ll 1$. The result will not be discussed in this paper.

For waves propagating in other directions, similar treatment can be followed to obtain the propagation constants.

6. CONCLUSIONS

The effective dielectric tensor and propagation constant for a plane wave in a random medium with anisotropic background are discussed in this paper. Approximate expressions are derived for a turbulent plasma in a static magnetic field. The elements of the effective dielectric tensor are found to depend on the magnitude as well as the direction of the wave vector **k**. There is a non-Hermitian part in this tensor, corresponding to the attenuation to the average fields in this medium due to random scattering of waves. The tensor reduces to the ordinary formula for magnetoionic medium when $\Delta X = 0$. The contributions to the propagation constants for ordinary and extraordinary modes are found to the order of $\langle(\Delta X)^2\rangle Y$. The new propagation constants depend on both the polar angle as well as the azimuth angle of the wave vector **k**. For the case of longitudinal propagation, the attenuation constants for the two modes are derived. They differ by terms of the order of $\langle(\Delta X)^2\rangle Y$. For a specific medium, for which the correlation function is assumed to be given, calculations are made for the propagation and attenuation constants. The results reduce to those of the special case treated by Keller and Karal[1] when $Y = 0$.

The present discussion is still limited to a weak random medium. For media with strong irregularities, extensions must be made in the formulation.

ACKNOWLEDGMENT

The author wishes to express his gratitude to Professor K. C. Yeh of the University of Illinois for many helpful suggestions and discussions during the course of this study.

APPENDIX A

In the following, inverse Fourier transforms used in Sec. 3 to derive the Green's function are given. The mean value theorems have been found useful in integrating over the surface of a p sphere:

$$\begin{aligned}
&\frac{1}{(2\pi)^3}\int d\mathbf{p}\,\frac{p_ip_j}{(p^2-k_1^2)^2}\,e^{i\mathbf{p}\cdot\mathbf{r}} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty \frac{p^2\,dp}{(p^2-k_1^2)^2}\iint \frac{p_ip_j}{p^2}\,e^{i\mathbf{p}\cdot\mathbf{r}}\,ds_p \\
&= \frac{r_ir_j}{2\pi^2r^2}\int_0^\infty \frac{dp}{(p^2-k_1^2)^2} \\
&\quad\times\left[\frac{p^3\sin pr}{r}+\frac{3p^2\cos pr}{r^2}-\frac{3p\sin pr}{r^3}\right] \\
&\quad-\frac{\delta_{ij}}{2\pi^2r}\int_0^\infty \frac{dp}{(p^2-k_1^2)^2}\left[\frac{p^2\cos pr}{r}-\frac{p\sin pr}{r^2}\right] \\
&= \frac{r_ir_j}{8\pi r^2}\left(ik_1-\frac{1}{r}\right)e^{ik_1r}+\frac{\delta_{ij}}{8\pi r}\,e^{ik_1r},
\end{aligned} \tag{A1}$$

$$\frac{1}{(2\pi)^3}\int d\mathbf{p}\,\frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{(p^2-k_1^2)^2}=\frac{i}{8\pi k_1}\,e^{ik_1r}. \tag{A2}$$

APPENDIX B

The tensor relation $C_{inmj}R_{nm}=B_{ij}$ used in Sec. 4 for any tensor R_{nm} is obtained by using Eqs. (2.1) and (4.3). They are given in the following:

$$\begin{aligned}
B_{11} &= C_{1nm1}R_{nm} = \langle(\Delta X)^2\rangle(R_{11}-iYR_{12}+iYR_{21}),\\
B_{12} &= C_{1nm2}R_{nm} = \langle(\Delta X)^2\rangle(R_{12}+iYR_{11}+iYR_{22}),\\
B_{13} &= C_{1nm3}R_{nm} = \langle(\Delta X)^2\rangle(R_{13}+iYR_{23}),\\
B_{21} &= C_{2nm1}R_{nm} = \langle(\Delta X)^2\rangle(R_{21}-iYR_{11}-iYR_{22}),\\
B_{22} &= C_{2nm2}R_{nm} = \langle(\Delta X)^2\rangle(R_{22}-iYR_{12}+iYR_{21}),\\
B_{23} &= C_{2nm3}R_{nm} = \langle(\Delta X)^2\rangle(R_{23}-iYR_{13}),\\
B_{31} &= C_{3nm1}R_{nm} = \langle(\Delta X)^2\rangle(R_{31}-iYR_{32}),\\
B_{32} &= C_{3nm2}R_{nm} = \langle(\Delta X)^2\rangle(R_{32}+iYR_{31}),\\
B_{33} &= C_{3nm3}R_{nm} = \langle(\Delta X)^2\rangle R_{33}.
\end{aligned} \tag{B1}$$

APPENDIX C

The first-order contributions to the propagation constants are given by (5.11) and (5.13) in terms of the following functions:

$$\begin{aligned}
\delta_1 &= Z_{11}\sin^2\theta\cos^2\phi+Z_{22}\sin^2\theta\sin^2\phi+Z_{33}\cos^2\theta\\
&\quad+(Z_{13}+Z_{31})\sin\theta\cos\phi\cos\theta+(Z_{23}+Z_{32})\\
&\quad\times\sin\theta\sin\phi\cos\theta+(Z_{21}+Z_{12})\\
&\quad\times\sin^2\theta\cos\phi\sin\phi;
\end{aligned} \tag{C1}$$

$$\begin{aligned}
\delta_2 &= [(1-X)(Z_{13}+Z_{31})+iXY(Z_{23}-Z_{32})]\\
&\quad\times\sin\theta\cos\phi\cos\theta+[(1-X)(Z_{23}+Z_{32})\\
&\quad+iXY(Z_{31}-Z_{13})]\sin\theta\sin\phi\cos\theta\\
&\quad+(1-X)(Z_{12}+Z_{21})\sin^2\theta\cos\phi\sin\phi\\
&\quad-(1-X)(Z_{22}+Z_{33})\sin^2\theta\cos^2\phi\\
&\quad-(1-X)(Z_{11}+Z_{33})\sin^2\theta\cos^2\phi\\
&\quad-[(1-X)(Z_{11}+Z_{22})+iXY(Z_{21}-Z_{12})]\\
&\quad\times\cos^2\theta+2(1-X)(Z_{11}+Z_{33}+Z_{22})\\
&\quad+iXY(Z_{21}-Z_{12});
\end{aligned} \tag{C2}$$

$$\begin{aligned}
\delta_3 &= (1-X)^2(Z_{11}+Z_{22}+Z_{33})\\
&\quad+i(1-X)XY(Z_{21}-Z_{12}).
\end{aligned} \tag{C3}$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Properties of Velocity-Dependent Potentials

ERASMO M. FERREIRA
Universidad Central de Venezuela, Caracas and
Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro

AND

NILO GUILLÉN AND JAVIER SESMA
Universidad Central de Venezuela, Caracas

(Received 25 January 1967)

Properties of the solutions of the Schrödinger equation with a velocity-dependent potential are studied. Particular attention is given to the examination of the singularities of the differential equation. In the particular cases of one dimension and of the $l = 0$ partial wave of a spherically symmetric problem, a simple correspondence is found between the velocity-dependent problem and a static one.

1. INTRODUCTION

VELOCITY-DEPENDENT potentials have been introduced and often used in the study of the interaction between nucleons.[1,2] The reason for this was that it was found that strong repulsion at short distances is a typical feature of the two-nucleon system. If experimental results on the nucleon–nucleon scattering experiments were to be explained by a potential model using static (i.e., velocity nondependent) terms only, a hard core had necessarily to be included among these terms. Such a highly singular potential, infinite inside a sphere of given radius, becomes cumbersome in calculations, especially if more complicated systems, such as a many-body system of nucleons, are to be treated. It was then shown that velocity-dependent potentials and static potentials with hard core can be equally good in describing the interaction of two nucleons.[3] A static hard core can thus be simulated by a term in the Schrödinger equation which depends on the momentum operator. This relation between velocity-dependent and hard-core potentials has deserved the attention of several authors.[4,5] The appropriate velocity-dependent potential may be such that it only introduces a well-behaved term in the Schrödinger differential equation, thus avoiding the high singularity that a hard-core potential would necessarily introduce. Thus, as far as the mathematical handling of the problem is concerned, a velocity-dependent potential may present advantages as compared with equivalent static potentials.

The description of the interaction of elementary particles by means of a Schrödinger equation with a potential is just an approximate, essentially nonrelativistic, idealization of a more fundamental and consistent description, which perhaps should be made in the framework of the theory of interacting quantized fields. This idealization by means of a potential model is suggested by electrodynamics and by gravitational theories, but the importance of any model is based primarily on its ability to agree with the facts of nature. Thus velocity-dependent potentials in the Schrödinger equation may provide models as good as the static ones for treating the interactions among particles in the nonrelativistic limit. The study of the properties of the Schrödinger equation modified by the presence of velocity-dependent terms thus seems to us to be of importance.

The presence of new terms with momentum operators acting on the wavefunction changes the form of the wave equation (by introducing terms with first-order derivatives of the wavefunction, for example). Thus, an examination of the properties of the new differential equation may be required, and this paper is devoted to this task. Since it seems that we are lacking a good understanding of the effect of velocity-dependent potentials in terms of the usual concepts of forces or, equivalently, of static potentials, an effort is also made in this direction.

2. THE WAVE EQUATION

Following Razavy, Field, and Levinger (Ref. 1), we introduce in the one-particle Schrödinger equation a "potential" of the form

$$V(\mathbf{r}, \mathbf{p}) = V_1(\mathbf{r}) - (\lambda/2m)\mathbf{p} \cdot J(\mathbf{r})\mathbf{p} \qquad (2.1)$$

---

[1] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962).

[2] A. M. Green, Nucl. Phys. **33**, 218 (1962); E. Werner, *ibid.* **35**, 324 (1962); F. Peischl and E. Werner, *ibid.* **43**, 372 (1963); M. Razavy, *ibid.* **50**, 465 (1964); B. H. J. McKellar, Phys. Rev. **134**, B1190 (1964); B. K. Srivastava, *ibid.* **133**, B545 (1964); **137**, B71, (1965); Nucl. Phys. **67**, 236 (1965); A. E. S. Green and R. D. Sharma, Phys. Rev. Letters **14**, 380 (1965); F. Tabakin and K. T. R. Davies, Phys. Rev. **150**, 793 (1966).

[3] O. Rojo and J. S. Levinger, Phys. Rev. **123**, 2177 (1961); O. Rojo and L. M. Simmons, *ibid.* **125**, 273 (1962).

[4] J. S. Bell, *The Many-Body Problem* (W. A. Benjamin, Inc., New York, 1962), p. 214.

[5] G. A. Baker, Phys. Rev. **128**, 1485 (1962); J. Tharrats, O. Cerceau, and O. Rojo, J. Math. Phys. **6**, 1315 (1965); R. M. May, Nucl. Phys. **62**, 177 (1965); S. Sunakawa and Y. Fukui, Progr. Theoret. Phys. (Kyoto) **34**, 693 (1965).

consisting of a static term and a strongly velocity-dependent part. $\mathbf{p}$ is the momentum operator $-i\hbar\nabla$, and in the above expression a scalar product is formed with its components. λ is a dimensionless quantity, and m is the mass of the particle. $V_1(\mathbf{r})$ and $J(\mathbf{r})$ are real functions of the position vector $\mathbf{r}$. The potential $V(\mathbf{r}, \mathbf{p})$ is Hermitian, parity conserving, and invariant under time reversal.[6]

Another form of velocity-dependent potential which has often been used is

$$U(\mathbf{r}, \mathbf{p}) = U_1(\mathbf{r}) - (\lambda/m)(p^2\omega(\mathbf{r}) + \omega(\mathbf{r})p^2).$$

This form is equivalent to Eq. (2.1) if the relations

$$J(\mathbf{r}) \equiv 4\omega(\mathbf{r}),$$
$$V_1(\mathbf{r}) - (\lambda\hbar^2/4m)\nabla^2 J(\mathbf{r}) \equiv U_1(\mathbf{r})$$

are satisfied. Then we have to discuss only one of these two forms, and we choose Eq. (2.1).

Since

$$\begin{aligned} V\psi &= V_1(\mathbf{r})\psi - (\lambda/2m)(-i\hbar\nabla)\cdot[J(\mathbf{r})(-i\hbar\nabla)\psi] \\ &= V_1(\mathbf{r})\psi + (\lambda\hbar^2/2m)[J(\mathbf{r})\nabla^2\psi + \nabla J(\mathbf{r})\cdot\nabla\psi], \end{aligned} \tag{2.2}$$

we obtain a "modified" Schrödinger equation where first-order derivatives of the wavefunction appear, namely

$$(-\hbar^2/2m)(1-\lambda J)\nabla^2\psi + (\lambda\hbar^2/2m)\nabla J\cdot\nabla\psi + V_1\psi = i\hbar\partial\psi/\partial t \tag{2.3}$$

or

$$(-\hbar^2/2m)\nabla\cdot[(1-\lambda J)\nabla\psi] + V_1\psi = i\hbar\partial\psi/\partial t. \tag{2.4}$$

Stationary-state wavefunctions of the form $\psi = \phi(\mathbf{r})e^{-iEt/\hbar}$ exist, with ϕ satisfying

$$(-\hbar^2/2m)\nabla\cdot[(1-\lambda J)\nabla\phi] + V_1\phi = E\phi. \tag{2.5}$$

"Plane-wave" solutions of the form $\phi = Ae^{i\mathbf{k}\cdot\mathbf{r}}$ exist in regions where $V_1(\mathbf{r}) = V_1 = \text{const}$, and $\nabla J = 0$, that is, where $J(\mathbf{r})$ is constant, the modulus of $\mathbf{k}$ being then given by

$$(\hbar^2/2m)(1-\lambda J)k^2 = E - V_1. \tag{2.6}$$

Let us now examine the properties of continuity of the derivatives of ϕ. Consider a volume element dv limited by $x_1 = x, x_2 = x + dx; y_1 = y, y_2 = y + dy; z_1 = z, z_2 = z + dz$. By multiplying Eq. (2.5) by dv, integrating over this element and using the divergence theorem, we get, if V_1 is continuous in this region,

$$\begin{aligned} -\frac{\hbar^2}{2m}\Bigg\{&\left(\left[(1-\lambda J)\frac{\partial\phi}{\partial x}\right]_{x_2} - \left[(1-\lambda J)\frac{\partial\phi}{\partial x}\right]_{x_1}\right) dy\,dz \\ &+ \left(\left[(1-\lambda J)\frac{\partial\phi}{\partial y}\right]_{y_2} - \left[(1-\lambda J)\frac{\partial\phi}{\partial y}\right]_{y_1}\right) dx\,dz \\ &+ \left(\left[(1-\lambda J)\frac{\partial\phi}{\partial z}\right]_{z_2} - \left[(1-\lambda J)\frac{\partial\phi}{\partial z}\right]_{z_1}\right) dx\,dy\Bigg\} \\ = (E - V_1)&\phi\,dx\,dy\,dz. \end{aligned} \tag{2.7}$$

Since dx, dy, dz are independent infinitesimals, we then get, when $x_1 \to x_2$,

$$\left[(1-\lambda J)\frac{\partial\phi}{\partial x}\right]_{x_1} = \left[(1-\lambda J)\frac{\partial\phi}{\partial x}\right]_{x_2}, \tag{2.8}$$

and the same for the other coordinates. Thus, $(1-\lambda J)\partial\phi/\partial n$ is continuous when we cross a surface orthogonal to an arbitrary vector $\mathbf{n}$, and consequently $(1-\lambda J)\nabla\phi$ is continuous, the same being true of $(1-\lambda J)\nabla\psi$.

Multiplying Eq. (2.4) from the left by ψ^*, and the corresponding complex-conjugate equation from the right by ψ, and subtracting the two expressions, we obtain

$$(-\hbar^2/2m)\nabla\cdot\{(1-\lambda J)(\psi^*\nabla\psi - \psi\nabla\psi^*)\} = i\hbar\partial(\psi^*\psi)/\partial t, \tag{2.9}$$

which has the form of a continuity equation

$$\nabla\cdot\mathbf{S} = -\partial P/\partial t$$

with the usual expression $P = \psi^*\psi$ for the density of probability, and a density of current

$$\mathbf{S} = (\hbar/2mi)(1-\lambda J)(\psi^*\nabla\psi - \psi\nabla\psi^*), \tag{2.10}$$

which differs from the usual expression by the factor $(1-\lambda J)$.

We have shown that the product $(1-\lambda J)\nabla\psi$ is continuous everywhere. P and $\mathbf{S}$ will then be continuous if ψ itself is continuous. The continuity of ψ is studied in the next section.

The Lagrangian function which, by a variational principle, leads to Eq. (2.4) is

$$\mathcal{L} = (1-\lambda J(\mathbf{r}))\nabla\psi^*\cdot\nabla\psi - (E - V_1)\psi^*\psi. \tag{2.11}$$

The commutator between $V(\mathbf{r}, \mathbf{p})$ and the angular-momentum operator is

$$\begin{aligned} [\mathbf{L}, V(\mathbf{r}, \mathbf{p})] &= i\hbar\,\mathbf{rot}(\mathbf{r}V_1(\mathbf{r})) \\ &\quad + i\hbar(-\lambda/2m)\mathbf{p}\cdot\{\mathbf{rot}(\mathbf{r}J(\mathbf{r}))\}\mathbf{p}, \end{aligned} \tag{2.12}$$

where in the last term the scalar product is between the two operators $\mathbf{p}$. If $V_1(\mathbf{r})$ and $J(\mathbf{r})$ are spherically

[6] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. 27, 281 (1941); S. Okubo and R. E. Marshak, Ann. Phys. (N.Y.) 4, 166 (1958).

symmetric, $V(r, p)$ will commute with all components of $\mathbf{L}$ and with L^2, and the wave equation will be separable in spherical coordinates.

3. CONTINUITY OF THE WAVEFUNCTION

Let us drop for a while the static potential $V_1(\mathbf{r})$ from our equation. From Eq. (2.3) we see that if $J(\mathbf{r})$ is continuous in a point, ψ and all its first and second derivatives will also be continuous there. We want to obtain information on what happens in a point where J may have a finite discontinuity. Let us consider that, in the neighborhood of a certain point, J depends on a single Cartesian coordinate x, and may be discontinuous on this variable. In other words, locally we have a one-dimensional wave equation

$$\frac{\partial\{[1 - \lambda J(x)]\partial\phi/\partial x\}}{\partial x} = -\frac{2m}{\hbar^2} E\phi. \qquad (3.1)$$

Let us call

$$\chi(x) = [1 - \lambda J(x)]\partial\phi/\partial x. \qquad (3.2)$$

We have already shown in Eq. (2.8) that $\chi(x)$ is continuous. The wave equation, Eq. (2.5), can be written

$$\partial\chi/\partial x = -(2mE/\hbar^2)\phi. \qquad (3.3)$$

By derivation of Eq. (3.3), we obtain, making use of Eq. (3.2),

$$[1 - \lambda J(x)]\, d^2\chi/dx^2 = -(2mE/\hbar^2)\chi. \qquad (3.4)$$

This can be written in the form of a usual Schrödinger equation for energy E

$$d^2\chi/dx^2 = (2m/\hbar^2)[U(x) - E]\chi \qquad (3.5)$$

with a potential

$$U(x) = -E\lambda J(x)/[1 - \lambda J(x)] \qquad (3.6)$$

which depends on the parameter E. Since in Eq. (3.5) no first derivatives occur, we have that, even if λJ suffers finite jumps, χ and $d\chi/dx$ are continuous (as in the usual Schrödinger equation). From Eq. (3.3), we then obtain that ψ is *continuous through a finite discontinuity in λJ.*

4. SINGULARITIES OF THE WAVE EQUATION

The "potential" $U(x)$ is singular wherever $\lambda J = 1$. Let us take Eq. (3.1) and investigate what happens with the wavefunction ϕ in such singular points. Let us suppose that $1 - \lambda J(x)$ passes through zero in a certain point x_0. If the first derivative of $1 - \lambda J$ in this point, $f'(x_0)$, is not zero, we have in the neighborhood of x_0,

$$1 - \lambda J(x) \approx (x - x_0)f'(x_0) + O((x - x_0)^2),$$

and Eq. (3.1) becomes, in this neighborhood, of the form

$$(x - x_0)\, d^2\phi/dx^2 + d\phi/dx = A\phi, \qquad (4.1)$$

where

$$A = 2mE/\hbar^2 f'(x_0). \qquad (4.2)$$

The general solution of this equation is a linear combination of a regular solution

$$\phi_1 = \sum_{n=0}^{\infty} a_n A^n (x - x_0)^n, \qquad (4.3)$$

and an irregular one, which is of the form

$$\phi_2 = \phi_1(x) \ln|A(x - x_0)| + \sum_{n=1}^{\infty} b_n A^n (x - x_0)^n. \qquad (4.4)$$

Thus, the general solution

$$\phi(x) = C_1\phi_1(x) + C_2\phi_2(x) \qquad (4.5)$$

behaves like $\ln|A(x - x_0)|$ when $x \to x_0$. This solution is acceptable since, in spite of increasing without limits as x approaches x_0, it is square integrable in this region (in fact, it is integrable even when raised to any finite power).

Thus, if $\lambda J(x)$ has a nonzero derivative when passing through the value 1, the wavefunction is defined by Eq. (4.5) in this region. For $x \approx x_0$, $\phi(x)$ in Eq. (4.5) is an even function of $(x - x_0)$, since the dominant terms give

$$\phi(x) \approx C_1 a_0 + C_2 a_0 \ln|A(x - x_0)|. \qquad (4.6)$$

Then, we can say that $\phi(x)$ is "continuous" when we cross x_0; that is, it assumes the same values when x tends to x_0 from either side.

If λJ jumps through the value 1, that is, if λJ is of the form of a step from a value lower than 1 to another one higher than 1, we can think of it as the limit of a linear increase, whose slope tends to infinity. Then, $A \to 0$, and in the interval (from x_1 to x_2, as shown in Fig. 1) in which the "jump" takes place, the wavefunction is well represented by Eq. (4.6). Since $\phi(x)$ in Eq. (4.6) is even and the solution must be continuous whenever $\lambda J \neq 1$, we have that its values at the right and left of the jump tend to be the same. We then conclude that the wavefunction is continuous through a finite jump of $\lambda J(x)$, even if this jump includes the singular value $\lambda J = 1$.

Let us suppose now that in the Taylor development of $1 - \lambda J(x)$ about its root x_0, the first derivative is also zero, that is,

$$1 - \lambda J(x) \approx (x - x_0)^2 f''(x_0)/2! + O((x - x_0)^3), \qquad (4.7)$$

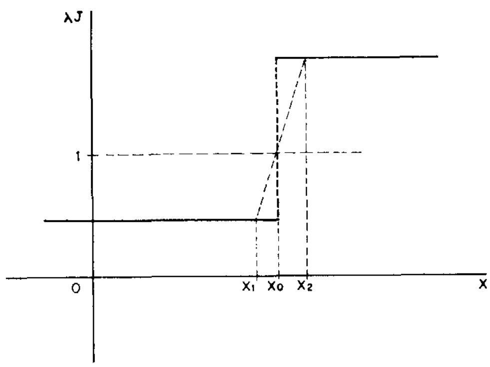


FIG. 1. A finite "jump" of the functions $\lambda J(x)$ may be considered as the limit of a linear increase, whose slope tends to infinity.

where $f''(x_0)$ is the second derivative of $1 - \lambda J(x)$ at the point x_0. The wave equation in the neighborhood of x_0 then becomes

$$(x - x_0)^2 \, d^2\phi/dx^2 + 2(x - x_0)\, d\phi/dx = B\phi, \quad (4.8)$$

where

$$B = -(2mE/\hbar^2)(2!/f''(x_0)), \quad (4.9)$$

and again x_0 is a regular singular point. The general solution of Eq. (4.8) is of the form

$$\phi = C_1(x - x_0)^{q_1} + C_2(x - x_0)^{q_2}, \quad (4.10)$$

with

$$q_1 = \tfrac{1}{2}(-1 + [1 + 4B]^{\frac{1}{2}})$$

and

$$q_2 = \tfrac{1}{2}(-1 - [1 + 4B]^{\frac{1}{2}}),$$

where by $[1 + 4B]^{\frac{1}{2}}$ we mean the square root of $1 + 4B$ whose real part is positive. Let us discuss the behavior of this solution in the several possible cases.

1. $B \geq 0$. Then, $q_2 < -1$ and to obtain a finite solution we must put $C_2 = 0$. Since $q_1 > 0$, we have $\phi(x) \to 0$ as x approaches x_0 from either side. Thus, ϕ is continuous in x_0, but the limit of the quotient of the values of ϕ in the two sides is not unity, since

$$\phi_R/\phi_L \xrightarrow[x \to x_0]{} (-1)^{q_1}, \quad (4.11)$$

where

$$\phi_R = \lim_{x \to x_0 + 0} \phi(x) \quad \text{and} \quad \phi_L = \lim_{x \to x_0 - 0} \phi(x).$$

Since in general q_1 is a nonintegral, there is a difference of phase between the values of the wavefunction in the two sides of x_0, and this difference of phase does not approach zero. On the other hand,

$$\phi' = C_1 q_1 (x - x_0)^{q_1 - 1} \quad (4.12)$$

and we have that

$$\phi' \xrightarrow[x \to x_0]{} 0 \quad \text{if} \quad q_1 - 1 > 0,$$

$$|\phi'| \xrightarrow[x \to x_0]{} \infty \quad \text{if} \quad q_1 - 1 < 0.$$

Since we have $\phi'/\phi \to q_1(x - x_0)^{-1}$, there comes

$$\frac{\phi_R'}{\phi_L'} \Big/ \frac{\phi_R}{\phi_L} = -1. \quad (4.13)$$

Thus, if the quotient of the values of the wavefunction on the two sides of x_0 tends to $\exp(i\delta)$, the quotient of its first derivatives tend to $\exp\{i(\delta + \pi)\}$.

On the other hand,

$$\chi \equiv (1 - \lambda J)\phi' \approx [f''(x_0)/2!] C_1 q_1 (x - x_0)^{q_1 + 1} \quad (4.14)$$

goes to zero on both sides of x_0, and is continuous [as we have seen in Eq. (2.8)]. The potential which appears in Eq. (3.5) is, in the neighborhood of x_0, given by

$$U(x) = -2!\, E/(x - x_0)^2 f''(x_0) = \hbar^2 B/2m(x - x_0)^2, \quad (4.15)$$

and will be repulsive in this case of $B > 0$. It corresponds to a function $\lambda J(x)$ which touches the line $\lambda J = 1$ in the point x_0; the curve of $\lambda J(x)$ has a minimum at this point in the case $E > 0$ and a maximum if $E < 0$.

For a given energy E, the larger the curvature radius of $1 - \lambda J$ at x_0 (that is, the smaller f''), the larger is the value of B. As B increases, the exponent q_1 becomes a larger positive number, and the wavefunction will tend to zero more rapidly (correspondingly, the ill-behaved part with exponent q_2 will become more divergent, since q_2 becomes more and more negative). Thus, for a flatter curve, the wavefunction tends to zero more rapidly. The potential in Eq. (3.6) then becomes more and more strongly repulsive.

As f'' reaches the value zero, the curve for $1 - \lambda J$ will present an inflection at x_0 if $f''' \neq 0$, and a

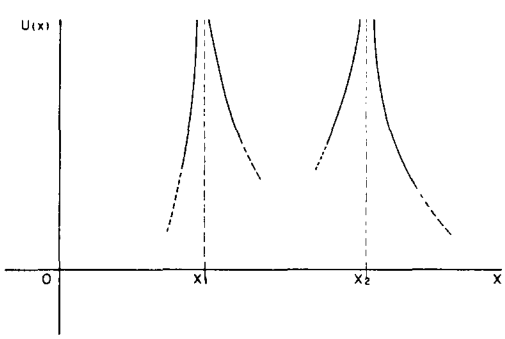


FIG. 2. The "static" potential $U(x)$, "equivalent" to a velocity-dependent one, presents two second-order singularities. We call it "static" because there are not in $U(x)$, defined by Eq. (3.6), terms depending on the linear momentum operator, although the energy E of the particle under consideration enters in the expression of $U(x)$ as a parameter.

maximum or minimum if $f''' = 0$ and $f^{\text{iv}} \neq 0$. The potential will be singular with a higher power of $x - x_0$; if this power is even, the potential is strongly repulsive or strongly attractive in the neighborhood of x_0; if the power is odd, it is repulsive in one side of x_0 and attractive in the other.

The fact that we have to put $C_2 = 0$ to obtain an acceptable physical wavefunction, implies that the existence of the second-order singularity at x_0 determines the wavefunction uniquely (up to a constant C_1). If there are two singular points of this kind, only for certain values of the energy E there would be a continuous physical solution for the differential equation extended over the region between the two points. This would determine eigenvalues for the problem. For $E > 0$, this case will happen when we have two successive contacts of $1 - \lambda J(x)$ with the value zero such that $f'' < 0$ (that is, $1 - \lambda J$ touches zero from below). For $E < 0$ we must have two contacts with zero from above. Both cases correspond to $U(x)$ being a singular repulsive potential at two different points (Fig. 2). Let us think in terms of this "equivalent" potential. According to Eq. (4.14), χ goes to zero in a singularity of the type given in Eq. (4.7). Then, as we know from the properties of the Schrödinger equation, a solution χ exists between two such points of second-order singularity if $U(x) < E$ somewhere between these two points. But in a situation such as that of Fig. 3, corresponding to $E < 0$, we have that between x_1 and x_2, $U(x)$ as given by Eq. (3.6) is always larger than E. Thus, there is no solution of negative energy for such a problem. In the case of $E > 0$, we have a situation as that of Fig. 4. As can be seen from Eq. (3.6) $U(x)$ is again larger than E everywhere between x_1 and x_2. Since solutions can exist at the right of x_2 and at the left of x_1, the region between these two points acts as a kind of *hard-core* potential.

2. $-\frac{1}{4} < B \leq 0$. The general solution of the differential equation is again of the form given in Eq. (4.10), with q_1 and q_2 having values in the intervals $-\frac{1}{2} < q_1 \leq 0$ and $-1 \leq q_2 < -\frac{1}{2}$. Since solutions behaving worse than $(x - x_0)^{-\frac{1}{2}}$ must be eliminated (because they are not square integrable), we must again put $C_2 = 0$. The accepted solution

$$\phi = C_1(x - x_0)^{q_1}$$

diverges as $x \to x_0$, but $\int |\phi|^2\, dx$ can be defined in any interval. We again have here that ϕ suffers a change of phase as we cross the singularity [it is multiplied by $(-1)^{q_1}$ as we pass from $x_0 - 0$ to $x_0 + 0$]. The modulus of the derivative $|\phi'|$ increases infinitely as we approach x_0, and Eq. (4.13) is still valid. Again

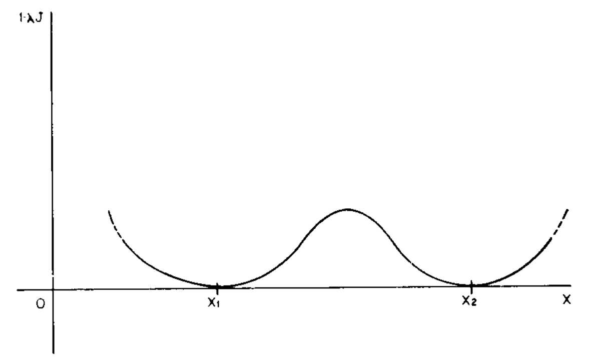


FIG. 3. The function $1 - \lambda J(x)$ touches, from above, the value zero in two different points. A particle, with mass m and energy $E < 0$, which suffers the velocity-dependent potential $(-\lambda/2m)p_x J(x) p_x$, "encounters" the "equivalent static" potential $U(x)$, given by Eq. (3.6) and shown in Fig. 2.

$\chi = (1 - \lambda J)\phi'$ is continuous as $1 - \lambda J$ passes through zero at the singularity.

The "equivalent" potential $U(x)$ is now attractive, since B is negative. But the wavefunction is not an oscillating function inside this infinitely deep potential well. This can be understood by remarking that the intensity B of the potential is determined by the value of the energy. With $B > -0.25$, for any given value of the energy, the range and depth of the potential will not be sufficient for the wavefunction to be able to oscillate. If we increase the value of the energy so as to tend to have a smaller wavelength and at the same time a broader region for the wavefunction to oscillate, the potential well becomes at the same time narrower and less deep.

3. $B < -\frac{1}{4}$. The general solution of Eq. (4.8) in the neighborhood of x_0 is of the form

$$\phi = D_1(x - x_0)^{-\frac{1}{2}+i\epsilon} + D_2(x - x_0)^{-\frac{1}{2}-i\epsilon}, \quad (4.16)$$

where $\epsilon = \frac{1}{2}|[1 + 4B]^{\frac{1}{2}}|$ assumes values ranging from zero to infinity. The wavefunction oscillates infinitely

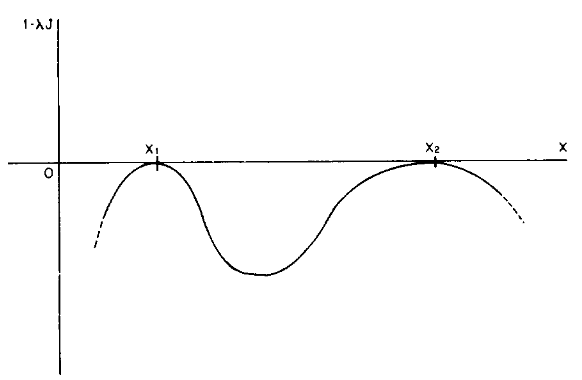


FIG. 4. The function $1 - \lambda J(x)$ touches, from below, the value zero in two different points. A particle, with mass m and energy $E > 0$, which suffers the velocity-dependent potential $(-\lambda/2m)p_x J(x) p_x$, "encounters" the "equivalent static" potential $U(x)$, given by Eq. (3.6) and shown in Fig. 2.

many times near x_0, and is not square integrable in this neighborhood. The expression

$$\chi = (1 - \lambda J)\phi' \approx [f''(x_0)/2!](x - x_0)^{\frac{1}{2}}[D_1(-\tfrac{1}{2} + i\epsilon)(x - x_0)^{i\epsilon} + D_2(-\tfrac{1}{2} - i\epsilon)(x - x_0)^{-i\epsilon}] \quad (4.17)$$

is continuous at the singularity, oscillating infinitely many times with decreasing amplitude as we approach x_0.

5. EFFECTIVE POTENTIALS: UNIDIMENSIONAL SQUARE WELLS AND BARRIERS

The first-order derivatives which appear in the wave equation (2.5) can be eliminated by a suitable change of function. Let us write

$$\eta = [1 - \lambda J]^{\frac{1}{2}}\phi. \quad (5.1)$$

We obtain in Eq. (2.5)

$$-(\hbar^2/2m)\nabla^2\eta + W\eta = E\eta, \quad (5.2)$$

with

$$W = (\hbar^2/2m[1 - \lambda J]^{\frac{1}{2}})\nabla^2([1 - \lambda J]^{\frac{1}{2}}) + (V_1 - E\lambda J)/(1 - \lambda J). \quad (5.3)$$

Equation (5.2) has the form of a Schrödinger equation with an effective potential W, in which the energy enters as a parameter. W is singular in points where $\lambda J = 1$, as the differential equation, Eq. (2.5), from which we started. If J has finite discontinuities, W will contain δ functions and first-order derivatives of δ functions, and these terms will complicate the analysis of the behavior of the solutions of Eq. (5.2). Thus, in spite of the simplicity of the relation between η and ϕ, Eq. (5.1), and of the apparent simplicity of Eq. (5.2), it seems that we do not gain very much from this transformation.

For the case of one-dimensional problems, the first-order derivatives of the wavefunction can be eliminated by introducing the auxiliary function $\chi(x)$ defined by Eq. (3.2), which obeys Eq. (3.5). Very simple relations exist between important properties of ϕ and χ. Excluding points for which $\lambda J = 1$, which were discussed in the previous section, χ and its derivatives of first and second order are continuous; physically acceptable solutions exist, for which these quantities are finite everywhere. By Eq. (3.2), the true wavefunctions, ϕ will also be finite in all space, and thus, physically acceptable. Let us see what occurs with the asymptotic behavior of these solutions.

If ϕ is a typical bound-state wavefunction, behaving as $e^{-\mu x}$ for large enough values of x (we suppose that λJ goes to zero at infinity, or at least is bounded there), then χ is also of the same form, representing also a bound-state wavefunction, with the same binding energy. On the other hand, if ϕ is a typical positive-energy wavefunction, behaving for large $|x|$ as

$$\phi \approx \sin(kx + \delta), \quad (5.4)$$

then χ will be

$$\chi \approx \sin(kx + \delta + \tfrac{1}{2}\pi), \quad (5.5)$$

with a phase shift increased by $\frac{1}{2}\pi$, and the same wavelength. Properties of the system described by the Schrödinger equation with velocity-dependent potential can then be deduced from the study of the Schrödinger equation with the static potential $U(x)$ given by Eq. (3.6) (the potential U is called static because momentum operators do not appear on it, although the energy enters in it as a parameter).

In a simple problem of square wells or barriers, with $\lambda J(x)$ being a constant λJ_0 in a certain interval and zero outside, the derivatives of λJ in the extremes of the interval introduce extra δ-function factors, which make the wave equation a little cumbersome to analyze. We have already used properties of the auxiliary function χ to study the properties of continuity of ϕ as we cross the walls of barriers and wells. The static potential $U(x)$ is zero outside and $U_0 = -E\lambda J_0/(1 - \lambda J_0)$ inside the interval.

For $E > 0$, the potential U_0 is repulsive for $\lambda J_0 > 1$ and for $\lambda J_0 < 0$, and attractive for $0 < \lambda J_0 < 1$. The opposite is true for negative energies. For $E < 0$ and $\lambda J_0 < 0$, the potential is attractive, but $U_0 > E$ and no bound-state solutions can exist since the energy is below the bottom of the well for all x. Bound states can then only exist for $\lambda J_0 > 1$.

As $\lambda J_0 \rightarrow 1$, $|U_0|$ increases without limit, U_0 becoming a hard core or an infinitely deep well. We have the case of scattering by a hard-core potential when $E > 0$ and $\lambda J_0 \rightarrow 1 + 0$. We have the case of scattering by an infinitely deep well when $E > 0$ and $\lambda J_0 \rightarrow 1 - 0$. When $E > 0$, as λJ_0 passes the value 1 from higher to lower values, the potential U changes suddenly from $+\infty$ to $-\infty$. We have bound states in an infinitely deep well when $E < 0$, $\lambda J_0 \rightarrow 1 + 0$.

In Eq. (3.1), we see that if $1 - \lambda J_0 = 0$ in an interval, we have $\phi = 0$ there. This is the typical behavior of the wavefunction in a hard-core problem: it equals zero in the walls and is zero inside the core. If $1 - \lambda J_0$ is very small negative, Eq. (3.1) has a violently oscillating solution, characteristic of the very deep well.

6. SPHERICALLY SYMMETRIC PROBLEMS: AN EQUIVALENT STATIC POTENTIAL FOR THE S-WAVE CASE

With V_1 and J in Eq. (2.1) depending only on the radial distance r, the wave equation can be separated in partial waves by putting

$$\phi(\mathbf{r}) = \sum_{l,m} R_l(r) Y_{l,m}(\theta, \varphi), \quad (6.1)$$

where $Y_{l,m}(\theta, \varphi)$ are the spherical harmonic functions, and the radial part satisfies the differential equation

$$(1 - \lambda J)\{R_l'' + (2/r)R_l' - [l(l+1)/r^2]R_l\} + [d(1 - \lambda J)/dr]R_l' + (k^2 - U_1)R_l = 0, \quad (6.2)$$

with $k^2 = 2mE/\hbar^2$, $U_1 = 2mV_1/\hbar^2$.

Due to the term $-\lambda[dJ(r)/dr][dR_l/dr]$, Eq. (6.2) cannot in general be written in the form of a usual Schrödinger equation. We now show how we can do it in the case of $l = 0$, with $V_1 = 0$. The s-wave radial equation is then

$$d[(1 - \lambda J)\, dR_0/dr]/dr + (2/r)(1 - \lambda J)\, dR_0/dr + k^2 R_0 = 0. \quad (6.3)$$

Let us introduce a new function

$$\chi_R = (1 - \lambda J)\, dR_0/dr. \quad (6.4)$$

We obtain

$$d\chi_R/dr + 2\chi_R/r + k^2 R_0 = 0. \quad (6.5)$$

Taking derivatives of this equation, and eliminating dR_0/dr by using Eq. (6.4), we get

$$\frac{d^2\chi_R}{dr^2} + \frac{2}{r}\frac{d\chi_R}{dr} + \left(\frac{k^2}{1 - \lambda J} - \frac{2}{r^2}\right)\chi_R = 0, \quad (6.6)$$

or

$$\frac{d^2\chi_R}{dr^2} + \frac{2}{r}\frac{d\chi_R}{dr} + [k^2 - U(r)]\chi_R = 0, \quad (6.7)$$

where

$$U(r) = 2/r^2 - k^2\lambda J(r)/[1 - \lambda J(r)]. \quad (6.8)$$

Now, Eq. (6.7) has the form of an s-wave radial Schrödinger equation with potential $(\hbar^2/2m)U(r)$, or alternatively, by considering $2/r^2 \equiv 1(1 + 1)/r^2$ as a centrifugal term, we can consider it as a p-wave radial equation with a simpler potential of assumed finite range,

$$\frac{\hbar^2}{2m} k^2\mu(r) \equiv -\frac{(\hbar^2/2m)k^2\lambda J(r)}{1 - \lambda J(r)}. \quad (6.9)$$

In terms of a reduced wavefunction

$$y_1(r) = r\chi_R(r) \quad (6.10)$$

Eq. (6.7) becomes

$$d^2y_1/dr^2 + k^2y_1 - k^2\mu(r)y_1 - (2/r^2)y_1 = 0. \quad (6.11)$$

Properties of the wavefunction for usual static potentials will then be valid for χ_R, and the behavior of R_0 can be obtained from Eq. (6.4) or Eq. (6.5). If R_0 is a bound-state wavefunction, that is, if it has the form $F(r)e^{-\alpha r}$ for large r [with (Fr) tending to a constant as $r \to \infty$], then χ_R has the form $G(r)e^{-\alpha r}$ for large r [where $G(r)$ tends to a constant; we assume that $\lambda J(r)$ becomes a constant for large r]. Thus, χ_R will also be the wavefunction for a bound state, with the same binding energy.

If $R_0(r)$ has an asymptotic behavior

$$R_0(r) \approx Ae^{ikr} + Be^{-ikr}, \quad (6.12)$$

then, the asymptotic behavior of χ_R is

$$\chi_R \approx Ae^{ikr} - Be^{-ikr}. \quad (6.13)$$

The phase shifts in the two problems differ only by a constant $\frac{1}{2}\pi$, that is, if

$$R_0 \approx (1/r)\sin(kr + \delta_0), \quad (6.14)$$

then

$$\chi_R \approx (1/r)\sin(kr + \delta_0 + \tfrac{1}{2}\pi). \quad (6.15)$$

Thus the values of the S-matrix element corresponding to $l = 0$ differ only by a sign in the two cases of the velocity-dependent potential and the equivalent static problem with potential U.

ACKNOWLEDGMENTS

We are indebted to Professor O. Rojo and Professor G. Beck for helpful discussions.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Substitution Group and Mirror Reflection Symmetry in Special Unitary Groups

S. J. ALIŠAUSKAS
Institute of Physics and Mathematics of the Academy of Sciences of Lithuanian S.S.R., Vilnius, Lithuanian S.S.R.

AND

A. P. JUCYS
V. Kapsukas State University Vilnius, Vilnius, Lithuanian S.S.R.

(Received 10 April 1967)

The substitutions leaving the character of the representation of the group SU_n invariant are considered. The phases induced by these substitutions on the basis functions are established. The substitution giving the contragrediency transformation has been found. This transformation is interpreted as the reflection of the subspace of commuting operators and the corresponding coordinate systems with respect to the rest subspace. The application of the substitution group to the resolution of multiplicity problem in the case of SU_3 is demonstrated.

I. INTRODUCTION

IN the theory of representations of the group SU_2, an important role[1] is played by the substitutions

$$j \to \bar{j} \equiv -j - 1, \tag{1a}$$

$$m \to \bar{m} \equiv -m, \tag{1b}$$

$j(j+1)$ and m being proportional to the proper values of the Casimir operator and of one of the infinitesimal operators, respectively. These substitutions are geometrically interpreted as the reflection of the coordinate system (1a) and that of the space (1b) with respect to the plane of the other two infinitesimal operators. This process is called mirror reflection. It brings the basis functions to their contragredient ones so that the invariant corresponding to the scalar product takes on the form

$$(|jm\rangle\,|jm\rangle) = \sum_m |jm\rangle\,|\bar{j}\bar{m}\rangle. \tag{2}$$

In generalizing[2] the procedure mentioned above to the group SU_3, instead of (1) one obtains the substitutions

$$\begin{aligned} \lambda &\to \bar{\lambda} = -\lambda - 2 \\ \mu &\to \bar{\mu} = -\mu - 2 \\ I &\to \bar{I} = -I - 1, \end{aligned} \tag{3a}$$

$$\begin{aligned} M &\to \bar{M} = -M \\ Y &\to \bar{Y} = -Y, \end{aligned} \tag{3b}$$

where $\lambda + \mu$ and μ are the lengths of two rows of the Young pattern characterizing the representation, I and M are quantum numbers of isospin (SU_2), and Y is the hypercharge. In this case the corresponding expression (2) is obtained by substitutions: $j \to \lambda\mu$, $\bar{j} \to \bar{\lambda}\bar{\mu}$, $m \to IMY$, $\bar{m} \to \bar{I}\bar{M}\bar{Y}$. Geometrically, (3a) is interpreted as the reflection of the coordinates of the weight space and (3b) as that of the weight space itself with respect to the remaining part of the space of the group.

The further attempts[3] to generalize these procedures by following the examples of G_2 and SU_3 resulted in revealing a whole group of substitutions that leave the character of the representation invariant. This group of substitutions has been found to be isomorphic to the Weyl group.

In view of usefulness of the aforementioned substitutions, it is expedient to generalize them to any special unitary group. The main aim of this paper is to present some results obtained by generalizing the substitution group to SU_n for any value of n. This question is considered in the next section.

As was stated by Ališauskas, Rudzikas, and Jucys,[3] the contragrediency transformation is very closely connected with the group of substitutions. For G_2 it is just one element of this group, and for SU_3 it is one element followed by the permutation of λ and μ. The contragrediency transformation in SU_n is considered in Sec. III. The geometrical interpretation of this contragrediency transformation, which is the generalization of the mirror reflection symmetry[1] of SU_2, is given there as well.

In the last section, by following the example of SU_3, we demonstrate the utility of the substitution

[1] A. P. Jucys and A. A. Bandzaitis, *The Theory of Angular Momentum in Quantum Mechanics* (Mintis, Vilnius, Lithuanian S.S.R., 1965).

[2] A. P. Jucys, A. V. Karosiene, and S. J. Ališauskas, JETP Pis'ma v Redaktsiyu, (1965) [English transl.: JETP Letters **1**, No. 4, 17 (1965)].

[3] S. J. Ališauskas, Z. B. Rudzikas, and A. P. Jucys, Doklady Akad. Nauk SSSR **172**, 58 (1967).

group for solving the multiplicity problem. By this another point of view will be given to the methods of attacking the multiplicity problem which are given recently by Baird and Biedenharn[4] and by Biedenharn, Giovannini, and J. D. Louck.[5]

II. SUBSTITUTION GROUP OF SU_n

The expression, as given by Weyl,[6] for the character of the representation of SU_n is

$$\chi(\lambda_1, \lambda_2, \cdots, \lambda_n) = |A_{kj}(n, k, \lambda_k, \varphi_j)|/|A_{kj}(n, k, \lambda_k, \varphi_j)|_{i_k=0}, \quad (4)$$

where

$$A_{kj} = \exp\{i[(n+1)/2 - k + \lambda_k]\varphi_j\}, \quad (5)$$

$$\lambda_k = m_{kn} - \sum_{i=1}^{n} \frac{m_{in}}{n}, \quad (6)$$

m_{kn} being the length of kth row of the Young pattern for the Weyl-basis tableau of the irreducible representation.

The permutation of rows of determinants in (4)

$$\begin{pmatrix} 1 & 2 & \cdots & k & \cdots & n \\ l_1 & l_2 & \cdots & l_k & \cdots & l_n \end{pmatrix} \quad (7)$$

is equivalent to the substitutions

$$m_{kn} \to m'_{kn} \equiv m_{l_k n} - l_k + k. \quad (8)$$

Transformations (8) constitute the group called the substitution group,[3] which is isomorphic to the Weyl group of SU_n and, obviously, isomorphic to the symmetric group S_n. The substitutions (8) have been found by Baird and Biedenharn[4] in examining the invariance of the Weyl dimension formula. It is obvious, of course, that the substitutions leaving invariant characters of irreducible representations do not affect the dimensions of these representations.

The elements of the substitution group (8) induce the similarity transformation on the representation. This unitary transformation induces the phase factors of basis functions. These basis functions are characterized by the Gelfand[7] pattern

$$(m) \equiv \begin{bmatrix} m_{1,n} \; m_{2,n} & \cdots & m_{n-1,n} \quad m_{n,n} \\ m_{1,n-1} \cdots m_{2,n-1} & \cdots & m_{n-1,n-1} \\ & \cdots & \\ & m_{1,1} & \end{bmatrix}, \quad (9)$$

where

$$m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j+1}. \quad (10)$$

It is seen that the substitutions (8) affect the first row of the Gelfand pattern only. It is evident that similar substitutions applied to the other rows of the Gelfand pattern do not affect the character of the irreducible representation, because they leave invariant the characters of the representations of corresponding subgroups SU_{n-z} $(z < n)$ of the group SU_n. It is the phases of basis functions which are affected by these substitutions. For this reason our task is to establish the rules according to which the substitutions

$$m_{ki} \to m'_{ki} \equiv m_{l_k i} - l_k + k \quad (k \leq i = 1, 2, \cdots, n) \quad (11)$$

change the phases of the basis functions.

We use Baird and Biedenharn[8] phase system. In this system, the matrix elements of the infinitesimal operators $E_{k,k-1}$ are positive. We require that they remained positive under all substitutions (11). At first we bring the formula (60) of Baird and Biedenharn[8] to the form

$$E_{k,k-1}|(m)\rangle = \sum_{i=1}^{k-1} A_i |(m)\rangle_{m_{i,k-1}\to m_{i,k-1}-1}, \quad (12)$$

$$A_i = \left[\frac{\prod_{j=1}^{i-1}(m_{j,k-2} - m_{i,k-1} - j + i)\prod_{j=i}^{k-2}(m_{i,k-1} - m_{j,k-2} + j - i)}{\prod_{j=1}^{i-1}(m_{j,k-1} - m_{i,k-1} - j + i)\prod_{j=i+1}^{k-1}(m_{i,k-1} - m_{j,k-1} + j - i)}\right]^{\frac{1}{2}} \times \left[\frac{\prod_{j=1}^{i}(m_{j,k} - m_{i,k-1} - j + i + 1)\prod_{j=i+1}^{k}(m_{i,k-1} - m_{j,k} + j - i - 1)}{\prod_{j=1}^{i-1}(m_{j,k-1} - m_{i,k-1} - j + i + 1)\prod_{j=i+1}^{k-1}(m_{i,k-1} - m_{j,k-1} + j - i - 1)}\right]^{\frac{1}{2}}. \quad (13)$$

In the Gelfand pattern on the right-hand side of (12), $m_{i,k-1}$ must be replaced by $m_{i,k-1} - 1$. It is worth noting, that all multipliers in (13) are made to be positive. After performing substitutions of the type (11), some of these multipliers turn into negative ones. These last must be made positive by taking out of the

[4] G. E. Baird and L. C. Biedenharn, J. Math. Phys. **5**, 1730 (1964).
[5] L. C. Biedenharn, A. Giovannini, and J. D. Louck, J. Math. Phys. **8**, 691 (1967).
[6] H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1939).
[7] G. E. Baird and L. C. Biedenharn, J. Math. Phys. **5**, 1723 (1964).
[8] G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963).

square root the imaginary unit i for each negative multiplier, taking into account its location (in numerator or denominator). The phase of the basis function must be such that, together with these imaginaries, gives a positive sign to the matrix element of $E_{k,k-1}$.

In formulating the rules found by which the exponent

$$\varphi = \sum m_{ij} \tag{14}$$

of the phase factor

$$\delta = (-1)^{\varphi} \tag{15}$$

is to be constructed, we use the term "permutation" instead of "substitution," because each substitution (11) brings the parameter m_{ij} into another position, the appearance of inhomogenous term $k - l_k$ being unessential. These rules are as follows.

I. When the permutation takes place in one, say the kth, row of the Gelfand pattern, then

(a) φ_k contains the parameters of adjacent rows (it is $k - 1$ and $k + 1$) only.

(b) φ_k is additive with respect to the cycles of permutation.

(c) For a given cycle of permutation in the kth row, φ_k contains $m_{i,k+1}$ and $m_{i-1,k-1}$, when the vertical line drawn through the positions of these parameters cross the permutation lines of one direction an odd number of times, the permutation line being an arrowed line going from the old position of m_{ik} to the new one.

(d) The presence of m_{in} in the phase factor may be neglected, because the phases are not defined with respect to these parameters, at least not until the phases of the Clebsch–Gordan coefficients are defined.

II. When permutation takes place in two or more rows of the Gelfand pattern, then

(a) At first φ is constructed according to I as the sum of contributions of each row subjected to permutation.

(b) If the permutation takes place in adjacent rows, the following corrections must be introduced:

Through each point taking part in permutation, draw straight vertical lines crossing adjacent rows on both sides (upper and lower ones). The parameter m_{ij} must be included in φ, in addition, if the vertical line drawn through the old position of m_{ij} crosses an odd number of times the permutation lines on two adjacent rows having arrows of the same direction as permutation line of m_{ij} and not crossed by the vertical line drawn through the new position of m_{ij}.

(c) The parameters m_{in} may be neglected in constructing the phase factor on the same grounds given in I.

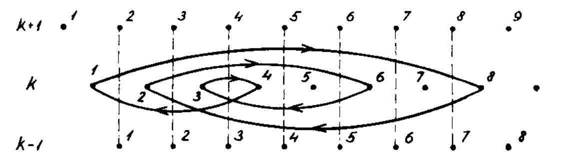


FIG. 1. Demonstration of the construction of phase factor, when the elements of kth row of the Gelfand pattern are permuted by the cycle (143628).

III. When permutation takes place in the Gelfand pattern with parameters already permuted, then the best way to construct the phase factor is this one:

(a) According to I and II, the normal Gelfand pattern is to be restored.

(b) The new permutation covering the first and second one is to be fulfilled according to I and II.

We illustrate I(c) and II(b) by examples. In Fig. 1, the three rows ($k - 1$, and $k + 1$) of the Gelfand pattern are represented. The first subscript of m_{ij} is indicated at each point of the diagram; the second subscript, labeling the rows (counting from the bottom of the pattern), is given on the left-hand side of the diagram. The transfer of the parameters by the cycle (143628) of permutation [substitution (11)] is indicated by full arrowed lines showing the directions of displacement of parameters. The broken vertical lines drawn through the positions of parameters $m_{i,k+1}$ and $m_{i-1,k-1}$ ($i = 1, 2, \cdots, 7$) crossing the permutation lines show that

$$\varphi_k = m_{1,k-1} + m_{3,k-1} + m_{6,k-1} + m_{7,k-1} + m_{2,k+1} + m_{4,k+1} + m_{7,k+1} + m_{8,k+1}. \tag{16}$$

Given in Fig. 2 are the same rows of the Gelfand pattern as in Fig. 1. Here we illustrate the construction of phase correction arising from the kth row because of the permutation being done in adjacent rows. This correction is

$$\Delta_k\varphi = m_{1,k} + m_{3,k} + m_{6,k}. \tag{17}$$

Here $m_{1,k}$ is present because the vertical line drawn through $m_{1,k}$ crosses the permutation line $1 \to 2$ on the row $k + 1$, which is not crossed by the line drawn through $m_{8,k}$ (the new position of $m_{1,k}$). $m_{2,k}$ is absent,

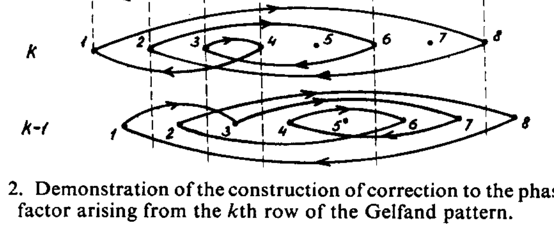


FIG. 2. Demonstration of the construction of correction to the phase factor arising from the kth row of the Gelfand pattern.

because, the vertical line drawn through its position crosses two permutation lines (one on each adjacent row) that are not crossed by the vertical line drawn through $m_{6,k}$. Further, for example, $m_{6,k}$ is present, because the corresponding vertical line crosses three permutation lines which are not crossed by the line drawn through $m_{3,k}$.

Concluding this section, we note that in the special case of SU_3, we have

$$\lambda = m_{13} - m_{23}, \qquad \mu = m_{23} \quad (m_{33} = 0),$$
$$I = (m_{12} - m_{22})/2, \quad M = m_{11} - (m_{12} + m_{22})/2,$$
$$Y = m_{12} + m_{22} - 2(m_{13} + m_{23})/3. \tag{18}$$

Then (11) gives the substitutions (2) of Ališauskas, Rudzikas, and Jucys.[3]

III. CONTRAGREDIENCY TRANSFORMATION

We seek a substitution that leads to a contragredient representation without any phase factor, like that done[1] in the case of SU_2. We find that this requirement is satisfied by the substitution

$$m_{ij} \rightarrow \bar{m}_{ij} = -m_{ij} + 2i - j - n. \tag{19}$$

In order to verify this statement, we must show that

$$\langle (m)| = |(m)\rangle^* = |(\bar{m})\rangle. \tag{20}$$

Here under $(\bar{m})$ is understood the Gelfand pattern in Eq. (9) in which all parameters are changed according to Eq. (19). If Eq. (20) is correct,

$$I = \sum_{m_{ij}(j \neq n)} |(m)\rangle\, |(\bar{m})\rangle \tag{21}$$

is invariant. This means that the result of operation by any infinitesimal operator must be zero; that is,

$$E_\alpha I = \sum_{m_{ij}(j\neq n)} \{\langle(\bar{m} + x)|\, E_\alpha\, |(\bar{m})\rangle\, |(\bar{m} + x)\rangle\, |(m)\rangle + \langle(m + x)|\, E_\alpha\, |(m)\rangle\, |(\bar{m})\rangle\, |(m + x)\rangle\} \tag{22}$$

must vanish. In order to visualize this in the second term in braces, we substitute $m_{ij} \rightarrow m_{ij} - x_{ij}$. Then we have

$$E_\alpha I = \sum_{m_{ij}(j\neq n)} \{\langle(\overline{m - x})|\, E_\alpha\, |(\bar{m})\rangle + \langle(m)|\, E_\alpha\, |(m - x)\rangle\}\, |(\overline{m - x})\rangle\, |(m)\rangle. \tag{23}$$

Taking $E_{k,k-1}$ and using (12), we find that the first matrix element in braces equals the second one with a minus sign, and thus (23) vanishes.

The substitution (19) can be carried out in two steps:

$$m_{ij} \rightarrow m_{j-i+1,j} + 2i - j - 1, \tag{24a}$$

$$m_{j-i+1,j} \rightarrow -m_{ij} - n + 1. \tag{24b}$$

The first step is the special case of (11) and the second one coincides up to a constant term with the substitution (11) of Baird and Biedenharn.[4] It is found that φ in (14) is additive for these substitutions and equals (up to terms containing m_{in})

$$\varphi_a = \varphi_b = \sum_{j=1}^{n-1} \sum_{i=1}^{j} m_{ij}. \tag{25}$$

This coincides with the phase of the conjugation operation of Baird and Biedenharn[7] with m_{in} omitted. Evidently φ_a and φ_b cancel one another and we obtain the contragrediency transformation without any phase multiplier, which was already pointed out.

It is easy to see that (24b) gives

$$M_k \equiv \frac{1}{k} \sum_{j=1}^{k} m_{jk} - \frac{1}{k+1} \sum_{j=1}^{k+1} m_{j,k+1} \rightarrow -M_k, \tag{26}$$

where M_k $(k = 1, 2, \cdots, n-1)$ are the proper values of the commuting infinitesimal operators (H_k). For this reason generalizing the geometrical interpretation of (1b) of SU_2, we interpret the substitution (24b) as the reflection of the weight space with respect to the rest part of the space. On the other hand, we interpret the substitution (24a) as the reflection of the coordinate system of the weight space with respect to the coordinate system of the remaining subspace. Consequently, the contragrediency transformation is to be interpreted as simultaneous reflection of the weight space and of corresponding coordinates with respect to the rest part of the space. This reflection is the generalization of mirror reflection symmetry considered by Jucys, Savukynas, and Bandzaitis[9] in the case of SU_2.

We can concentrate our attention on the subspace of commuting operators only, because quantities corresponding to these operators can only take-on definite values. The above-stated reflection is the simultaneous inversion of this space and of a corresponding coordinate system. On this occasion, attention must be called to the possibility of an alternative interpretation of the symmetries connected with the quantities corresponding to the commuting operators.

We are going now to give a geometrical interpretation of the Young pattern for the Weyl basis tableau for contragredient representation. It is seen from Eq. (19) that all parameters are negative in contragrediently transformed states. It is possible to make the tableau a lexical one by a suitable choice of the diagram. Such a diagram is shown in Fig. 3. The left part of this diagram is obtained from the normal Weyl basis tableau by reflection with respect to the central line. This part of the diagram is lexical when read from right to left. The right-hand part of this

[9] A. P. Jucys, A. J. Savukynas, and A. A. Bandzaitis, Liet. Fiz. Rinkinys 5, 171 (1965).

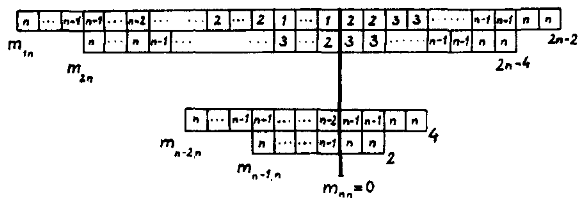


FIG. 3. The Weyl basis tableau for the contragredient representation of SU_n.

diagram represents nonhomogenous terms of (19). Lengths of these rows are $2n-2$, $2n-4$, $\cdots$. This part of the pattern does not take part in symmetry operations. When going to the subgroup SU_{n-1}, it is necessary to take away all squares filled with the index n on the left- and right-hand sides from the central line. Subsequent shrinkage of the diagram is obtained by removing the antisymmetrical part of the remaining diagram in order to go over from U_{n-1} to SU_{n-1}.

In the special case of SU_3, (18) and (19) give the substitutions (3). It is evident that (24a) corresponds to (3a), and (24b) to (3b).

IV. THE MULTIPLICITY PROBLEM FOR SU_3

Limiting ourselves to the group SU_3, we show how the substitution group can be used for the solution of multiplicity problem. Our demonstration is founded on the requirement of invariance of the coupling scheme with respect to the substitutions (11). This means that the additional quantum number in the Clebsch–Gordan coefficient must be conserved under these substitutions.

In a manner similar to that used in the case of SU_2 by Jucys, Savukynas, Bandzaitis, Karosiene, and Našlenas[10] for isoscalar factors of SU_3, we obtain relations

$$\begin{bmatrix}(\lambda\mu) & (\lambda'\mu') & (\lambda+a,\mu+b)_\rho \\ IY & I'Y' & I+k, Y+Y'\end{bmatrix} \tag{27a}$$

$$= \sum_{\rho'} \delta^A_{\rho\rho'}(-1)^{2I'}\begin{bmatrix}(\mu,-\lambda-\mu-3) & (\lambda'\mu') & (\mu+b,-\lambda-\mu-3-a-b)_{\rho'} \\ IY & I'Y' & I+k, Y+Y'\end{bmatrix} \tag{27b}$$

$$= \sum_{\rho'} \delta^B_{\rho\rho'}(-1)^{2I'}\begin{bmatrix}(-\lambda-\mu-3,\lambda) & (\lambda'\mu') & (-\lambda-\mu-3-a-b,\lambda+a)_{\rho'} \\ IY & I'Y' & I+k, Y+Y'\end{bmatrix} \tag{27c}$$

$$= \sum_{\rho'} \delta^K_{\rho\rho'}(-1)^{2I'}\begin{bmatrix}(-\mu-2,-\lambda-2) & (\lambda'\mu') & (-\mu-2-b,-\lambda-2-a)_{\rho'} \\ IY & I'Y' & I+k, Y+Y'\end{bmatrix} \tag{27d}$$

$$= \sum_{\rho'} \delta^L_{\rho\rho'}(-1)^{\frac{\lambda'-\mu'}{3}+\frac{Y'}{2}-k}\begin{bmatrix}(\lambda+\mu+1,-\mu-2) & (\lambda'\mu') & (\lambda+\mu+1+a+b,-\mu-2-b)_{\rho'} \\ IY & I'Y' & I+k, Y+Y'\end{bmatrix} \tag{27e}$$

$$= \sum_{\rho'} \delta^M_{\rho\rho'}(-1)^{\frac{\lambda'-\mu'}{3}+\frac{Y'}{2}+k}\begin{bmatrix}(-\lambda-2,\lambda+\mu+1) & (\lambda'\mu') & (-\lambda-2-a,\lambda+\mu+1+a+b)_{\rho'} \\ IY & I'Y' & I+k, Y+Y'\end{bmatrix} \tag{27f}$$

$$= (-1)^{a+b-\frac{2(\lambda'-\mu')}{3}+\frac{Y'}{2}-I'}\begin{bmatrix}(\lambda\mu) & (\lambda'\mu') & (\lambda+a,\mu+b)_\rho \\ \bar{I}Y & I'Y' & \bar{I}-k, Y+Y'\end{bmatrix}. \tag{28}$$

Here $\delta_{\rho\rho'}$ is a unitary matrix. In (28) it is absent because λ and μ do not change in any way in this case. By the way, it must be mentioned that (28) is obtained on the supposition that in the rules given in Sec. II, the parameters m_{in} are included.

Now our problem is to find a definition of Clebsch–Gordan coefficients such that $\delta_{\rho\rho'}$ in Eq. (27) will be diagonal. We observe that this condition is fulfilled by the isoscalar factors for $(\lambda'\mu') = (11)$ as given by Hecht[11] and Kuryan, Lurié, and Macfarlane[12] in the form of algebraic expressions.

Let us construct an operator of the form

$$T^{(\lambda'\mu')} = P(10)^{\lambda'-r}(01)^{\mu'-r}(11)^u_1(11)^{r-u}_2. \tag{29}$$

where (10), (01), and (11) represent operators $T^{(10)}$,

[10] A. P. Jucys, A. J. Savukynas, A. A. Bandzaitis, A. V. Karosiene, and E. P. Našlenas, Liet. Fiz. Rinkinys 4, 173 (1964).

[11] K. T. Hecht, Nucl. Phys. 62, 1 (1965).

[12] J. G. Kuriyan, D. Lurié, and A. J. Macfarlane, J. Math. Phys. 6, 722 (1965).

$T^{(01)}$, and $T^{(11)}$, respectively, the last one being from the direct product of two infinitesimal operators. For this reason two kinds appear. Equation (29) is the direct product with maximal weight, which is symbolized by P. Such a coupling can be realized with the help of the formulas of "stretched" Clebsch–Gordan coefficients given by Sharp and von Baeyer[13] and Ponzano.[14] It may be pointed out that the multiplicity of the representation does not exceed

$$\min\,(r+1,\lambda+1,\mu+1,\lambda+a+1,\mu+b+1) \tag{30}$$

and that r is

$$\min\left(\lambda',\mu',\frac{2\lambda'+\mu'-2a'-b'}{3},\frac{\lambda'+2\mu'-a'-2b'}{3}\right), \tag{31}$$

where a', b' are homogenous parts of substitutions (2) of Ališauskas, Rudzikas, and Jucys[3] which one obtains from (11), as was mentioned at the end of Sec. II, and which are used in (27).

The matrix elements of the operator (29) with maximal value of I' are proportional to the Clebsch–Gordan coefficients

$$\begin{bmatrix} (\lambda\mu) & (\lambda'\mu') & (\lambda+a,\mu+b)_u \\ IY & \tfrac{1}{2}(\lambda'+\mu')Y' & I+\tfrac{1}{2}(\lambda'+\mu')-u+c, Y+Y' \end{bmatrix}, \tag{32}$$

which vanish unless c is a nonpositive integer. u and $u-c$, where

$$0 \le u,\, u-c \le r, \tag{33}$$

label the Clebsch–Gordan coefficients with all other parameters fixed. This gives the triangular matrix of the form

$$\begin{array}{c|cccccc} {}_{u}\diagdown^{u-c} & 0 & 1 & 2 & \cdots & r-1 & r \\ \hline 0 & x & x & x & & x & x \\ 1 & 0 & x & x & & x & x \\ 2 & 0 & 0 & x & & x & x \\ \cdots & & & & & & \\ r-1 & 0 & 0 & 0 & & x & x \\ r & 0 & 0 & 0 & & 0 & x. \end{array} \tag{34}$$

When the multiplicity is less than $r+1$, the corresponding number of upper rows of (34) are to be filled by zeros.

It may be shown that the triangular character of (34) is not spoiled by substitutions (11). Consequently, $\delta_{\rho\rho'}$ must be diagonal and because of reality and orthonormality of the Clebsch–Gordan coefficients they are of the form

$$\delta^X_{\rho\rho} = (-1)^{\varphi_X}. \tag{35}$$

For the phase system of Baird and Biedenharn[8] in which the Clebsch–Gordan coefficients with $(\lambda'\mu') = (10)$ or (01) and $I' = 0$ are taken to be positive, one obtains

$$\begin{aligned} \varphi_A &= \varphi_B = 0, \\ \varphi_K &= \lambda' + \mu' + a + b + r - u, \\ \varphi_L &= \varphi_M = r - u. \end{aligned} \tag{36}$$

It must be noted that in our method of resolution of multiplicity problems, contrary to the method of conjugation operation of Baird and Biedenharn,[4] there is no difficulty in the case where $\lambda' \neq \mu'$. Nevertheless, some inconvenience arises from the fact that in Clebsch–Gordan coefficients defined by this method u, being conserved under transposition of the first and third columns, is not conserved when the second column is subjected to the transposition. However, this fact causes no harm because such a transposition can be avoided in practical applications.

[13] R. T. Sharp and H. von Baeyer, J. Math. Phys. 7, 1105 (1966).
[14] G. Ponzano, Nuovo Cimento 41A, 142 (1966).

Canonical Realizations of the Rotation Group

M. PAURI
Istituto di Fisica dell'Università, Parma, and Istituto Nazionale di Fisica Nucleare, Gruppo di Parma, Parma, Italy

AND

G. M. PROSPERI
Istituto di Fisica dell'Università, Bari, and Istituto Nazionale di Fisica Nucleare, Sottosezione di Bari, Bari, Italy

(Received 12 January 1967)

A general theory of the realizations of Lie groups by means of canonical transformations in classical mechanics, proposed in a preceding paper, is applied to the rotation group. A number of significant physical examples corresponding to *nonirreducible* realizations is treated in detail: specifically, the mass point, the rotator, and the rigid body with a fixed point. The explicit form of the possible *irreducible* realizations is worked out. Such realizations do not directly correspond to any realistic physical model but play a relevant role for the introduction of the spin in classical mechanics.

I. INTRODUCTION

In a preceding paper[1] a general theory has been proposed to characterize all the possible realizations of the finite Lie groups by means of canonical transformations in a classical phase space. In the present paper we want to apply the general theory given there to the rotation group, discussing some fundamental physical examples.[2,3]

Let us summarize the most relevant results of the first paper.

First of all, assuming we have a physical system characterized by the canonical coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$, we defined as a canonical realization $\mathfrak{R}(\mathbf{a})$ of a Lie group $\mathcal{G}$ a set of transformations

$$q'_i = q'_i(q_1, \cdots, q_n, p_1, \cdots, p_n \mid a_1, \cdots, a_r),$$
$$p'_j = p'_j(q_1, \cdots, q_n, p_1, \cdots, p_n \mid a_1, \cdots, a_r), \quad (1)$$
$$(i, j = 1, \cdots, n)$$

[where $\mathbf{a} \equiv (a_1, \cdots, a_r)$ are the parameters of $\mathcal{G}$] homomorphic to $\mathcal{G}$ and leaving the Poisson brackets among the fundamental dynamical variables invariant. Besides, we considered the Lie algebra of $\mathcal{G}$

$$[X_\rho, X_\sigma] = c^\tau_{\rho\sigma} X_\tau \quad (\rho, \sigma = 1, \cdots, r), \quad (2)$$

and we showed that the functions $y_\rho(q, p)$ which generate the infinitesimal transformations of the realization $\mathfrak{R}$ satisfy the following Poisson bracket relations:

$$\{y_\rho, y_\sigma\} = c^\tau_{\rho\sigma} y_\tau + d_{\rho\sigma}, \quad (3)$$

where the $d_{\rho\sigma}$'s are constants depending on the particular realization and satisfying the conditions

$$d_{\rho\sigma} + d_{\sigma\rho} = 0, \quad c^\tau_{\rho\sigma} d_{\tau\lambda} + c^\tau_{\lambda\rho} d_{\tau\sigma} + c^\tau_{\sigma\lambda} d_{\tau\rho} = 0. \quad (4)$$

The number of independent $d_{\rho\sigma}$'s can be reduced to a minimum, say s, which is characteristic of the group $\mathcal{G}$, by means of a substitution of the form $y_\rho \to y_\rho + \gamma_\rho$ (γ_ρ constants).

Then, we proved (see Ref. 1, Theorem 1) that, considering first the generators $y_\rho(q, p)$ as independent variables, it is possible to construct r independent functions of the y_ρ's: $\mathfrak{Q}_1(y), \cdots, \mathfrak{Q}_h(y)$, $\mathfrak{P}_1(y), \cdots, \mathfrak{P}_h(y)$, $\mathfrak{I}_1(y), \cdots, \mathfrak{I}_k(y)$ satisfying

$$\begin{aligned} \{\mathfrak{Q}_i, \mathfrak{Q}_j\} &= \{\mathfrak{P}_i, \mathfrak{P}_j\} = \{\mathfrak{Q}_i, \mathfrak{I}_t\} = \{\mathfrak{P}_j, \mathfrak{I}_t\} \\ &= \{\mathfrak{I}_t, \mathfrak{I}_{t'}\} = 0, \\ \{\mathfrak{Q}_i, \mathfrak{P}_j\} &= \delta_{ij}; \quad i, j = 1, \cdots, h; \\ & \qquad t, t' = 1, \cdots, k, \quad 2h + k = r, \end{aligned} \quad (5)$$

which can be ordered therefore within the following scheme, hereafter referred to as the *Scheme A*:

$$\begin{array}{ll} \mathfrak{P}_1(y) \cdots \mathfrak{P}_h(y) & \mathfrak{I}_1(y) \cdots \mathfrak{I}_k(y) \\ \mathfrak{Q}_1(y) \cdots \mathfrak{Q}_h(y) & \text{-----------} . \end{array} \quad (6)$$

Here the Poisson bracket between any two expressions is minus one if they are on the same column and zero otherwise. The expressions $\mathfrak{I}(y)$ which actually are the only independent functions of the generators which have zero Poisson brackets with all of them, have been called the *canonical invariants*. Their number k is given by the formula

$$k = r - \text{generic rank} \, \|c^\tau_{\rho\sigma} y_\tau + d_{\rho\sigma}\|. \quad (7)$$

[1] M. Pauri and G. M. Prosperi, J. Math. Phys. 7, 366 (1966).
[2] An independent formalism for constructing infinitesimal canonical realizations in the specific case of orthogonal and unitary algebras has recently been developed by S. Titeica, D. H. Constantinescu, and V. Florescu. We thank these authors for having sent us preprints.
[3] S. Ţiţeica, *Representation of the Infinitesimal Unitary Groups by Infinitesimal Canonical Transformations*, I.F.A.-FT-59 (1965); S. Ţiţeica, D. H. Constantinescu, and V. Florescu, *Representation of the Infinitesimal Orthogonal Groups by Infinitesimal Canonical Transformations*, I.F.A.-FT-61 (Bucharest, Romania, 1966).

Now, if we replace the variables y_ρ by their actual expressions in terms of a system of canonical variables (q, p) of a given canonical realization $\mathfrak{R}$, the expressions $\mathfrak{J}(y_1, \cdots, y_r)$ will not result in general independent functions of the canonical variables. This means that a certain number, say $k - l$, of relations of the form

$$f_\alpha(\mathfrak{J}_1, \cdots, \mathfrak{J}_k) = \text{const}, \quad \alpha = 1, \cdots, k - l, \quad (8)$$

may possibly exist. Let us call $\mathfrak{J}_1', \cdots, \mathfrak{J}_l'$ $(l \leq k)$ the independent $\mathfrak{J}$ (or functions of them) and let us denote by $\mathfrak{J}_{l+1}', \cdots, \mathfrak{J}_k'$ those functions of the $\mathfrak{J}$ which turn out to be identically equal to constants. Then, we proved a second fundamental theorem (see Ref. 1, Theorem 2) which can be stated as follows: given any canonical realization $\mathfrak{R}$ in terms of the variables (q, p), it is possible to perform a suitable fixed (with respect to the group parameters) canonical transformation in the phase space of the system, defining new canonical variables

$$Q_i = Q_i(q, p), \quad P_j = P_j(q, p); \quad i, j = 1, \cdots, n, \quad (9)$$

such that:

(1) $Q_1, \cdots, Q_h, P_1, \cdots, P_h$ coincide ordinately with $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_h, \mathfrak{P}_1, \cdots, \mathfrak{P}_h$;

(2) a set of variables $P_{h+1}, \cdots, P_{h+l}$ coincide ordinately with $\mathfrak{J}_1', \cdots, \mathfrak{J}_l'$ while, obviously, $\mathfrak{J}_{l+1}', \cdots, \mathfrak{J}_k'$ have zero Poisson brackets with all the variables Q_i, P_j.

This result can be summarized in the following scheme which is referred to as *Scheme B*:

I	II	III	IV
$P_1 \equiv \mathfrak{P}_1 \cdots P_h \equiv \mathfrak{P}_h$	$P_{h+1} \equiv \mathfrak{J}_1' \cdots P_{h+l} \equiv \mathfrak{J}_l'$	$\mathfrak{J}_{l+1}' \cdots \mathfrak{J}_k'$	$P_{h+l+1} \cdots P_n$
$Q_1 \equiv \mathfrak{Q}_1 \cdots Q_h \equiv \mathfrak{Q}_h$	$Q_{h+1} \cdots Q_{h+l}$		$Q_{h+l+1} \cdots Q_n$

(10)

where, again, the Poisson bracket is minus one when between any two expressions shown in the same column and zero otherwise. The form that the realization $\mathfrak{R}$ assumes in terms of the variables just introduced has been called the *typical form.*

From the structure of Scheme B it is apparent that:

(1) the *invariants* appearing in the third set are numerical constants, as we already know;

(2) the generators $y_\rho = y_\rho(Q, P)$ are actually functions only of the variables of the first set and of $P_{h+1}, \cdots, P_{h+l}$;

(3) the variables of the fourth set are left unchanged along with the *invariants* $P_{h+1}, \cdots, P_{h+l}$ under the group transformations;

(4) the variables $Q_{h+1}, \cdots, Q_{h+l}$, under an infinitesimal transformation [see Ref. 1, Eqs. (4)], change according to the simple law

$$Q_{h+u}' = Q_{h+u} - \delta a^\tau \frac{\partial y_\tau(Q_1, \cdots, Q_h, P_1, \cdots P_{h+l})}{\partial P_{h+u}};$$

$$u = 1, \cdots l, \quad (11)$$

while the most significant transformation properties are those of the variables of the first set.

The classification of the possible canonical realizations of $\mathcal{G}$ is obtained in terms of:

(1) the values of the constants $d_{\rho\sigma}$'s (after reduction to their minimum number);

(2) the number and the actual expression of the *invariants* appearing in the second and in the third set; and the values assumed by the *invariants* of the third set;

(3) the number of the variables of the fourth set, which will be called henceforth *inessential variables.*

Finally we recall that the realizations for which the second and the fourth sets are empty have been defined as transitive or *irreducible.* Such realizations are characterized by the property that no manifold $\mathcal{F}(q, p) = \text{const}$ there exists in the phase space, which is left invariant by the transformations of the group $\mathfrak{R}$. All other realizations have been called intransitive or *nonirreducible.*

In the second section of the paper we discuss the reduction (actually elimination) of the constants $d_{\rho\sigma}$ in the case of the rotation group, and the construction of Scheme A. All the nontrivial realizations are obviously faithful and no significant subcase arises. In Sec. 3 we discuss the *typical form* for a number of interesting examples, namely the mass point, the rotator and the rigid body with a fixed point. In Sec. 4 the *irreducible* canonical realizations of the rotation group are explicitly constructed. They are connected with the introduction of the spin in classical mechanics. Finally, in Sec. 5, the relations between particular canonical realizations and the transformation properties of the elementary spinor are discussed.

The complicated developments are confined in the appendixes.

The paper intends to be introductory to the discussion of the canonical realizations of the Galilei group.

2. GENERALITIES AND SCHEME A

The Lie algebra of the rotation group $O^+(3)$ is

$$[\mathcal{M}_i, \mathcal{M}_j] = \epsilon_{ijk}\mathcal{M}_k; \quad i, j, k = x, y, z, \tag{12}$$

where $\mathcal{M}_x$, $\mathcal{M}_y$, and $\mathcal{M}_z$ are the infinitesimal operators of the rotations around the x, y, z axes, respectively.

Let us denote with M_x, M_y, and M_z the corresponding generators of the three infinitesimal rotations in a given canonical realization. According to Eqs. (3), we have

$$\{M_i, M_j\} = \epsilon_{ijk}M_k + d_{ij}. \tag{13}$$

Taking advantage of the fact that the generators are defined up to an additive constant, we perform the substitution

$$M_i \to M_i + \tfrac{1}{2}\epsilon_{irs}d_{rs} \tag{14}$$

and obtain

$$\{M_i, M_j\} = \epsilon_{ijk}M_k. \tag{15}$$

Thus, in the case of the rotation group, all the constants d_{ij} can be eliminated.

Now, let us construct the variables of Scheme A. First, let us put, for instance, $\mathfrak{P} \equiv M_z$. Then we look for a function $\mathfrak{Q} = \mathfrak{Q}(M_x, M_y, M_z)$ such that

$$\{\mathfrak{Q}, \mathfrak{P}\} \equiv \frac{\partial \mathfrak{Q}}{\partial M_x}\{M_x, M_z\} + \frac{\partial \mathfrak{Q}}{\partial M_y}\{M_y, M_z\} = 1,$$

that is,

$$M_x\frac{\partial \mathfrak{Q}}{\partial M_y} - M_y\frac{\partial \mathfrak{Q}}{\partial M_x} = 1. \tag{16}$$

The general solution of Eq. (16) is

$$\mathfrak{Q} = \arctan(M_y/M_x) + f(M_x^2 + M_y^2), \tag{17}$$

where f is an arbitrary function of its argument. Assuming $f \equiv 0$ in Eq. (17), and observing that the expression

$$M^2 = M_x^2 + M_y^2 + M_z^2$$

has a zero Poisson bracket with all the generators, we arrive at Scheme A:

$$\begin{array}{ll} \mathfrak{P} = M_z & \mathfrak{J} = M^2 \\ \mathfrak{Q} = \arctan(M_y/M_x) & \text{———}. \end{array} \tag{18}$$

3. THE TYPICAL FORM FOR SOME SIGNIFICANT EXAMPLES

At this point the first step of the general program is completed. We want now to construct, for some significant examples, the fundamental canonical transformation (9) which leads to the variables of the *typical form* (Scheme B). In the present section we discuss the cases of the mass point, the rotator, and the rigid body with fixed point. In treating such examples, the following philosophy is adopted: We assume the physical system to be characterized by a set of configurational coordinates q^i and by the expression of its kinetic energy $T = \frac{1}{2}a_{ij}(q)\dot{q}^i\dot{q}^j$. The transformation properties of the configurational coordinates $q'^i = q'^i(q, \mathbf{a})$ are assumed to be obviously defined by the nature of the physical system. Consequently, the transformations of the generalized velocities $\dot{q}^i$ are

$$\dot{q}^{i'} = (\partial q'^i/\partial q^j)\dot{q}^j, \tag{19}$$

while those of the conjugate momenta $p_i = \partial T/\partial \dot{q}^i = a_{ij}\dot{q}^j$ are the contragradient ones

$$p_i' = (\partial q^j/\partial q'^i)p_j. \tag{20}$$

(i) Mass point. As configurational coordinates, let us adopt the Cartesian ones, x, y, z. The conjugate momenta are $p_x = m\dot{x}$, $p_y = m\dot{y}$, $p_z = m\dot{z}$. In this case, the transformation properties of the q_x, q_y, q_z and of the p_x, p_y, p_z coincide separately with those of the transformation group itself. Assuming the passive point of view, the infinitesimal transformation under a rotation $\delta\omega$ around the z axis can be written as:

$$\begin{cases} q_x' = q_x + \delta\omega q_y, \\ q_y' = q_y - \delta\omega q_x, \\ q_z' = q_z, \end{cases} \quad \begin{cases} p_x' = p_x + \delta\omega p_y, \\ p_y' = p_y - \delta\omega p_x, \\ p_z' = p_z. \end{cases} \tag{21}$$

Then, the corresponding canonical generator M_z is easily found to be

$$M_z = q_x p_y - q_y p_x. \tag{22}$$

Parallel expressions are deduced for M_x and M_y. In vector notations

$$\mathbf{M} = \mathbf{q} \times \mathbf{p}. \tag{23}$$

Thus the generators of the rotations coincide with the angular momentum, a well-known result! (see for instance Ref. 4, Chap. IV).

In order to construct the variables of the typical form it is convenient to introduce polar coordinates. One has

$$\begin{aligned} q_x &= r\sin\theta\cos\varphi, \\ q_y &= r\sin\theta\sin\varphi, \\ q_z &= r\cos\theta, \\ p_x &= \sin\theta\cos\varphi\cdot p_r - \frac{\sin\varphi}{r\sin\theta}p_\varphi + \frac{\cos\varphi\cos\theta}{r}p_\theta, \\ p_y &= \sin\theta\sin\varphi\cdot p_r + \frac{\cos\varphi}{r\sin\theta}p_\varphi + \frac{\sin\varphi\cos\theta}{r}p_\theta, \\ p_z &= \cos\theta\cdot p_r - \frac{\sin\theta}{r}p_\theta. \end{aligned} \tag{24}$$

[4] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959).

Then it follows

$$M_x = -\sin\varphi\cdot p_\theta - \cos\varphi\cot\theta\cdot p_\varphi,$$
$$M_y = \cos\varphi\cdot p_\theta - \sin\varphi\cot\theta\cdot p_\varphi, \quad (25)$$
$$M_z = p_\varphi,$$

and

$$M^2 = p_\theta^2 + (1/\sin^2\theta)p_\varphi^2. \quad (26)$$

Now, let us consider Scheme B. Since there are clearly variables such as φ, θ, p_θ having nonzero Poisson bracket with M^2, we must necessarily have in Scheme B two variables Q_1, P_1 in the first set, two variables Q_2, P_2 in the second one and two variables Q_3, P_3 in the fourth one, while the third set is empty. Furthermore, since the variables r and p_r have zero Poisson bracket with the three generators, we can directly put $Q_3 = r$, $P_3 = p_r$. The variables $Q_1 \equiv \mathfrak{Q}$, $P_1 \equiv \mathfrak{P}$ and $P_2 \equiv \mathfrak{J}$ can be obtained in terms of φ, θ, p_φ, p_θ by inserting Eqs. (25) into Eqs. (18). We are, then, left with the construction of the variable Q_2. To this aim we have to search first for a function $g(\varphi, \theta, p_\varphi, p_\theta)$ such that

$$\begin{aligned}\{Q_1, g\} &= 0,\\ \{P_1, g\} &= 0.\end{aligned} \quad (27)$$

The second equation gives at once $\partial g/\partial\varphi = 0$ and the first explicitly becomes

$$(p_\theta^2\sin^2\theta + p_\varphi^2\cos^2\theta)\frac{\partial g}{\partial p_\varphi} - p_\theta p_\varphi\frac{\partial g}{\partial p_\theta} + p_\varphi\sin\theta\cos\theta\frac{\partial g}{\partial\theta} = 0. \quad (28)$$

Two possible independent solutions of Eq. (28) are

$$\begin{cases} g_1 \equiv M^2 = p_\theta^2 + (1/\sin^2\theta)p_\varphi^2 \quad \text{and} \\ g_2 = p_\theta\tan\theta. \end{cases} \quad (29)$$

Every other solution has to be a function of these. Then Q_2 must be also a function of g_1 and g_2. By imposing the condition $\{Q_2, M^2\} = 1$, it follows that

$$Q_2 = \frac{1}{2M}\arctan\frac{p_\theta\tan\theta}{M}, \quad (M = |\mathbf{M}|) \quad (30)$$

(see Appendix A). The Scheme B for the mass point realization is completed.

The generators of the infinitesimal transformations, when expressed in terms of the variables Q, P, assume the form

$$M_x = (P_2 - P_1^2)^{\frac{1}{2}}\cos Q_1,$$
$$M_y = (P_2 - P_1^2)^{\frac{1}{2}}\sin Q_1, \quad (31)$$
$$M_z = P_1.$$

Then the transformations of the variables Q_1, P_1, P_2 for a given infinitesimal rotation $\delta\omega$ are

$$\begin{aligned} Q_1' &= Q_1 + \delta\omega_x\frac{P_1}{(P_2 - P_1^2)^{\frac{1}{2}}}\cos Q_1 + \delta\omega_y\frac{P_1}{(P_2 - P_1^2)^{\frac{1}{2}}}\sin Q_1 - \delta\omega_z,\\ P_1' &= P_1 - \delta\omega_x(P_2 - P_1^2)^{\frac{1}{2}}\sin Q_1 + \delta\omega_y(P_2 - P_1^2)^{\frac{1}{2}}\cos Q_1,\\ Q_2' &= Q_2 - \delta\omega_x\frac{1}{2(P_2 - P_1^2)^{\frac{1}{2}}}\cos Q_1 - \delta\omega_y\frac{1}{2(P_2 - P_1^2)^{\frac{1}{2}}}\sin Q_1, \end{aligned} \quad (32)$$

while, as we repeat once more, the variables P_2, Q_3, and P_3 remain unchanged.

The finite transformations can be obtained by expressing Q_1, P_1, and Q_2 in terms of Cartesian coordinates and conjugate momenta. Alternately, the same result can be obtained by integrating directly the system (32). In this connection, we observe that the expressions $\delta P_1 \equiv P_1' - P_1$, $\delta Q_1 \equiv Q_1' - Q_1$, $\delta Q_2 \equiv Q_2' - Q_2$ are functions only of the variables Q_1 and P_1 (and of P_2 as a parameter), not of Q_2. Then, we stress the point that, once the system formed by the two first equations, which is a closed one, is solved, the problem of the construction of the finite expression for Q_2 is reduced to a simple quadrature (see Sec. 4 and Appendix B). This is true in general; see Ref. 1, Sec. 3. Using as parameters for the rotations the Euler angles α, β, γ (we adopt throughout the conventions used in Ref. 4), the result is

$$\begin{aligned} Q_1' &= \arctan\left[\cos\beta\cdot\tan(Q_1+\alpha) + \frac{\sin\beta\cdot P_1}{\cos(Q_1+\alpha)\cdot(P_2 - P_1^2)^{\frac{1}{2}}}\right] - \gamma,\\ P_1' &= \cos\beta\cdot P_1 - \sin\beta\cdot\sin(Q_1+\alpha)\cdot(P_2 - P_1^2)^{\frac{1}{2}},\\ Q_2' &= Q_2 - \frac{1}{2P_2^{\frac{1}{2}}}\arctan\left[\frac{\tan\beta\cdot\sin Q_1\cdot\cos Q_1\cdot(P_2 - P_1^2) - \cos Q_1\cdot P_1(P_2 - P_1^2)^{\frac{1}{2}}}{\tan\beta\cdot P_1P_2 + \sin Q_1\cdot P_2(P_2 - P_1^2)^{\frac{1}{2}}}\right]. \end{aligned} \quad (33)$$

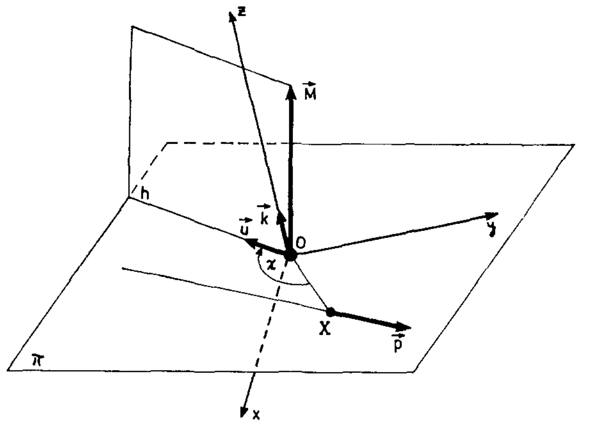


FIG. 1. The geometrical meaning of the variable Q_2 for the mass point realization: $\chi = 2MQ_2 = \arctan(p_\theta \tan\theta/M)$.

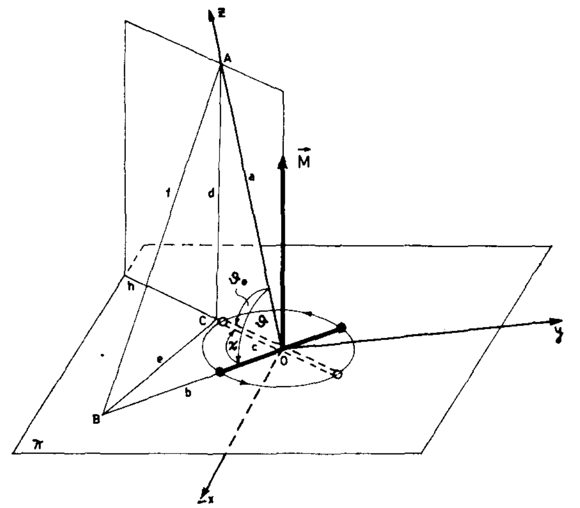


FIG. 2. The geometrical meaning of the variable Q_2 for the realization corresponding to the rotator: Assuming $a = 1$ it follows that $d = \sin\theta_0$, $c = \cos\theta_0$, $b = \cos\theta_0/\cos\chi$, $e = \cos\theta_0 \tan\chi$, $f = (\sin^2\theta_0 + \tan^2\chi\cos^2\theta_0)^{\frac{1}{2}}$. Applying the Carnot theorem to the triangle OAB: $f^2 = a^2 + b^2 - 2ab\cos\theta$, it results that $\cos\chi = \cos\theta/\cos\theta_0$. Then, since $\cos\theta_0 = (M^2 - M_z^2)^{\frac{1}{2}}/M$, it follows that [cf. Eq. (26)]

$$\chi = 2MQ_2 = \arctan\frac{(M^2\sin^2\theta - p_\varphi^2)^{\frac{1}{2}}}{M\cos\theta} = \arctan(p_\theta\tan\theta/M).$$

We return later to the problem of the integration of the first two equations (see Sec. 4).

In order to understand the geometrical meaning of the variable Q_2, let us consider a unit vector $\hat{n}$ and a quantity Θ so that $\{\Theta, \mathbf{M}\cdot\hat{n}\} = 1$. Thus, for a rotation ω around the direction of $\hat{n}$ it follows $\Theta' = \Theta + \omega$. In this sense Θ can be interpreted as an angular coordinate in a plane orthogonal to $\hat{n}$. Now, from $\{Q_2, M^2\} = 1$ we have also $\{2MQ_2, M\} = 1$; then, being $M = \mathbf{M}\cdot\mathbf{M}/M$, $2MQ_2$ has to stand for an angular coordinate in the plane orthogonal to $\mathbf{M}$. In fact, let us call X the mass point, O being the origin of the coordinate system and π being the plane through O which is orthogonal to $\mathbf{M}$. Obviously X belongs to π. One can easily see that $2MQ_2$ coincides with the angle χ between OX and the intersection h of π with the half-plane from $\mathbf{M}$ which contains the positive z axis (see Fig. 1). To this aim, it is sufficient to observe that the unit vector $\hat{u}$ along Oh is of the form $\hat{u} = a\hat{k} + b\mathbf{M}$, where $\hat{k}$ is the unit vector along the positive z axis. Then, since $\mathbf{M}\cdot\hat{u} = 0$ and normalizing, it follows that

$$\hat{u} = (M^2\hat{k} - M_z\mathbf{M})/M(M^2 - M_z^2)^{\frac{1}{2}}$$

and consequently

$$\cos\chi = \hat{u}\cdot(\mathbf{r}/r) = \frac{M\cos\theta}{(M^2 - M_z^2)^{\frac{1}{2}}}, \quad (\mathbf{r} \equiv \overrightarrow{\mathrm{OX}}),$$

whence, using Eqs. (25) and (26), the relation $\chi = 2MQ_2$ is readily obtained.

(ii) Rotator. The configurational variables of the system may be specified by the two polar angles φ, θ. Then the kinetic energy has the form

$$T = \tfrac{1}{2}I(\dot\theta^2 + \sin^2\theta\dot\varphi^2)$$

and the conjugate momenta result

$$p_\varphi = \frac{\partial T}{\partial\dot\varphi} = I\sin^2\theta\dot\varphi, \quad p_\theta = \frac{\partial T}{\partial\dot\theta} = I\dot\theta,$$

being I the moment of inertia of the system. Such a model can be considered to be deduced from the mass point by imposing the constraint $r = \text{const}$ within the configuration space. The transformation properties of φ, θ, p_φ, and p_θ are then the same as for the homonymous variables of the previous case, so that the generators of the three rotations are given again by Eqs. (25) and (26). The expressions of Q_1, P_1, P_2, and Q_2 for the rotator are consequently the same as for the mass point, the only difference between the two cases being the missing of the variables Q_3, P_3 in Scheme B for the former one. The geometrical meaning of the quantity $2MQ_2$ in this case is simply that of providing the angle formed by the rotator with the half-plane from the angular momentum $\mathbf{M}$ through the positive z axis (see Fig. 2 and the geometrical derivation indicated there).

(iii) Rigid body with a fixed point. The configuration of the system can be now characterized by the three Euler angles φ, θ, ψ. The kinetic energy assumes the form

$$T = \tfrac{1}{2}I_1\bar\Omega_{\bar x}^2 + \tfrac{1}{2}I_2\bar\Omega_{\bar y}^2 + \tfrac{1}{2}I_3\bar\Omega_{\bar z}^2,$$

where the $\bar\Omega_{\bar i}$'s ($\bar i = \bar x, \bar y, \bar z$) are the angular velocity components referred to the body system

$$\begin{aligned}\bar\Omega_{\bar x} &= \cos\psi\dot\theta + \sin\theta\sin\psi\dot\varphi,\\ \bar\Omega_{\bar y} &= -\sin\psi\dot\theta + \sin\theta\cos\psi\dot\varphi,\\ \bar\Omega_{\bar z} &= \dot\psi + \cos\theta\dot\varphi.\end{aligned} \tag{34}$$

The conjugate momenta are

$$p_\varphi = \sin\theta\sin\psi\cos\psi\cdot(I_1 - I_2)\dot\theta + (I_1\sin^2\theta\sin^2\psi + I_2\sin^2\theta\cos^2\psi + I_3\cos^2\theta)\dot\varphi + I_3\cos\theta\cdot\dot\psi,$$
$$p_\theta = (I_1\cos^2\psi + I_2\sin^2\psi)\dot\theta + \sin\theta\sin\psi\cos\psi\cdot(I_1 - I_2)\dot\varphi,$$
$$p_\psi = I_3\dot\psi + I_3\cos\theta\cdot\dot\varphi. \tag{35}$$

The transformation properties of φ, θ, ψ in terms of the Euler angles α, β, γ corresponding to a certain rotation, are expressed by (see Appendix C)

$$\varphi' = \arctan\frac{\sin(\varphi-\alpha)}{\cos\beta\cos(\varphi-\alpha) - \sin\beta\cot\theta} - \gamma,$$
$$\theta' = \arccos[\cos\beta\cos\theta + \sin\beta\sin\theta\cos(\varphi-\alpha)],$$
$$\psi' = \psi + \arctan\frac{\sin(\varphi-\alpha)}{\cos\theta\cos(\varphi-\alpha) - \cot\beta\sin\theta}. \tag{36}$$

From these formulas, the finite transformation properties for the momenta could in principle be deduced according to Eqs. (19) and (20). We do not give them explicitly. Let us consider instead the infinitesimal transformations. In terms of the rotation angle $\delta\boldsymbol{\omega}$,[5] the transformations of the configurational variables are

$$\varphi' = \varphi + \delta\omega_x\sin\varphi\cot\theta - \delta\omega_y\cos\varphi\cot\theta - \delta\omega_z,$$
$$\theta' = \theta - \delta\omega_x\cos\varphi - \delta\omega_y\sin\varphi,$$
$$\psi' = \psi - \delta\omega_x\frac{\sin\varphi}{\sin\theta} + \delta\omega_y\frac{\cos\varphi}{\sin\theta}. \tag{37}$$

Consequently the momenta transform according to

$$p'_\varphi = p_\varphi - \delta\omega_x\left(\cos\varphi\cot\theta\cdot p_\varphi + \sin\varphi\cdot p_\theta - \frac{\cos\varphi}{\sin\theta}p_\psi\right) - \delta\omega_y\left(\sin\varphi\cot\theta\cdot p_\varphi - \cos\varphi\cdot p_\theta - \frac{\sin\varphi}{\sin\theta}p_\psi\right),$$
$$p'_\theta = p_\theta + \delta\omega_x\left(\frac{\sin\varphi}{\sin^2\theta}p_\varphi - \frac{\sin\varphi\cos\theta}{\sin^2\theta}p_\psi\right) - \delta\omega_y\left(\frac{\cos\varphi}{\sin^2\theta}p_\varphi - \frac{\cos\varphi\cos\theta}{\sin^2\theta}p_\psi\right), \tag{38}$$
$$p'_\psi = p_\psi.$$

From Eqs. (38) it can be verified that the generators of the infinitesimal transformations are the angular momentum components in the space system. As a matter of fact, they are

$$M_x = \cos\varphi\cdot p_\theta + (\sin\varphi/\sin\theta)p_\psi - \sin\varphi\cot\theta\cdot p_\varphi,$$
$$M_y = \sin\varphi\cdot p_\theta - (\cos\varphi/\sin\theta)p_\psi + \cos\varphi\cot\theta\cdot p_\varphi,$$
$$M_z = p_\varphi, \tag{39}$$

so that

$$M^2 = p_\theta^2 + (\sin^2\theta)^{-1}(p_\varphi^2 + p_\psi^2) - 2(\cos\theta/\sin^2\theta)p_\varphi p_\psi. \tag{40}$$

The same result, however, could be simply deduced by observing that the rigid body with a fixed point could be considered to be derived from a system of n mass points by introducing suitable constraints and that for this last system it can be immediately verified [cf. Case (i)] that the generators are just given by the components of the total angular momentum referred to the space system

$$\mathbf{M} = \sum_{i=1}^{n}\mathbf{q}_i \times \mathbf{p}_i.$$

It is of interest for us to consider also the expressions of the angular momentum components referred to the body system. They are

$$\bar M_{\bar x} = \cos\psi\cdot p_\theta + (\sin\psi/\sin\theta)p_\varphi - \sin\psi\cot\theta\cdot p_\psi,$$
$$\bar M_{\bar y} = \sin\psi\cdot p_\theta - (\cos\psi/\sin\theta)p_\varphi + \cos\psi\cot\theta\cdot p_\psi,$$
$$\bar M_{\bar z} = p_\psi. \tag{41}$$

Let us point out that Eqs. (41) are formally identical to Eqs. (39) except for the substitutions $\varphi \rightleftarrows \psi$, $p_\varphi \rightleftarrows p_\psi$. It can be shown furthermore that the $\bar M_{\bar i}$'s ($\bar i = \bar x, \bar y, \bar z$) have zero Poisson brackets with the M_i's and that they satisfy the same Poisson bracket relations [cf. Eqs. (15)]. It can also be easily verified that *the $\bar M_{\bar i}$'s can be thought of as the generators of the infinitesimal transformations of the second parameter group of the rotation group as the M_i's are of the first one.*

Turning our attention to Scheme B, we note that, as in Case (i), there must be two inessential variables Q_3, P_3. They can be constructed by referring to Scheme A [Eqs. (18)] and replacing there the $\bar M_{\bar i}$'s for the M_i's. Thus we put

$$P_3 = \bar M_{\bar z}, \quad Q_3 = \arctan(\bar M_{\bar y}/\bar M_{\bar x}).$$

It remains to specify the variable Q_2 conjugated to $P_2 \equiv \mathfrak{I} = M^2$. This variable Q_2 cannot depend on φ and ψ since it has to have a zero Poisson bracket with $P_1 = p_\varphi$ and $P_3 = p_\psi$. The condition for a function $g(\theta, p_\varphi, p_\theta, p_\psi)$ to have zero Poisson brackets also with Q_1 and Q_3 is expressed by the following system

[5] As it is well known, the Euler angles are not convenient to parametrize the transformations in the neighborhood of the identity since the identity itself is a singular point for such a parametrization and the Euler angles cease to be essential parameters there.

of differential equations:

$$\left[p_\theta^2 + \left(\frac{p_\psi}{\sin\theta} - \cot\theta \cdot p_\varphi\right)^2\right]\frac{\partial g}{\partial p_\varphi} + \left[\frac{1}{\sin\theta}\left(\cot\theta \cdot p_\psi - \frac{p_\varphi}{\sin\theta}\right)p_\theta\right]\frac{\partial g}{\partial p_\theta} - \left[\frac{p_\psi}{\sin\theta} - \cot\theta \cdot p_\varphi\right]\frac{\partial g}{\partial\theta} = 0,$$

$$\left[p_\theta^2 + \left(\frac{p_\varphi}{\sin\theta} - \cot\theta \cdot p_\psi\right)^2\right]\frac{\partial g}{\partial p_\psi} + \left[\frac{1}{\sin\theta}\left(\cot\theta \cdot p_\varphi - \frac{p_\psi}{\sin\vartheta}\right)p_\theta\right]\frac{\partial g}{\partial p_\theta} - \left[\frac{p_\varphi}{\sin\theta} - \cot\theta \cdot p_\psi\right]\frac{\partial g}{\partial\theta} = 0. \qquad (42)$$

Two independent solutions of such a system are

$$\bar{g}_1 \equiv M^2 = p_\theta^2 + (\sin^2\theta)^{-1}(p_\varphi^2 + p_\psi^2) - 2(\cos\theta/\sin^2\theta)p_\varphi p_\psi \qquad (43)$$

and

$$\bar{g}_2 = \frac{M^2\cos\theta - p_\varphi p_\psi}{[(M^2 - p_\varphi^2)(M^2 - p_\psi^2)]^{\frac{1}{2}}}. \qquad (44)$$

Then, as for Case (i), the variable Q_2 is easily obtained in the form

$$Q_2 = \frac{1}{2M}\arctan\frac{p_\theta\tan\theta}{M - (p_\varphi p_\psi/M\cos\theta)} \qquad (45)$$

(see Appendix D). The geometrical meaning of the expression $2MQ_2$ is that of providing the rotation angle of the body around the direction of the angular momentum $\mathbf{M}$, precisely the angle defined by the half-planes from $\mathbf{M}$ through the intrinsic and through the fixed z axes (see Fig. 3; a geometrical derivation is also sketched there).

The results concerning Scheme B for the three examples dealt with, are summarized in Table I.

4. THE IRREDUCIBLE REALIZATIONS

In all the examples dealt with above, the invariant $\mathfrak{I} = M^2$ appears itself as a canonical variable and thus it does not have a definite value. Moreover, in the Cases (i) and (iii) there occur two inessential variables. According to the definitions given in Ref. 1, all the corresponding realizations are clearly *nonirreducible* (intransitive). The phase-space manifolds defined by equations of the form

$$\mathcal{F}(P_2, Q_3, P_3) = \text{const} \quad \text{or} \quad \mathcal{F}(P_2) = \text{const}$$

are in fact invariant manifolds. Actually, one can easily be convinced that none of the most intuitive

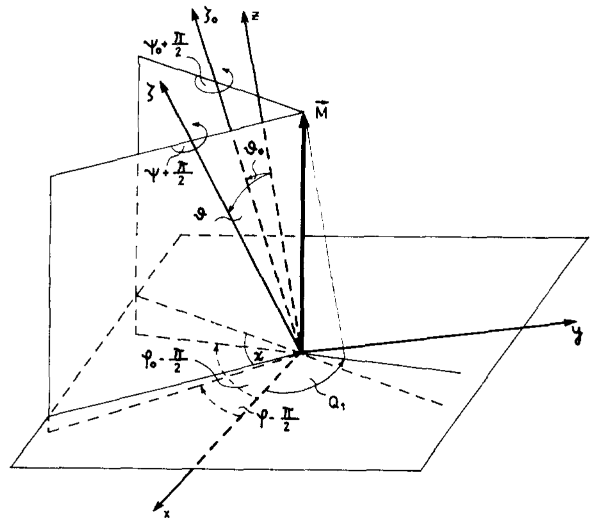


FIG. 3. The geometrical meaning of the variable Q_2 for the rigid body realization: The geometrical derivation can be obtained in the simplest way by using a procedure based on spinor calculus. By expressing a "transformed" spinor

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta\exp[-\frac{1}{2}i(\varphi+\psi)] \\ -i\sin\frac{1}{2}\theta\exp[\frac{1}{2}i(\varphi-\psi)] \end{pmatrix}$$

in terms of an initial spinor

$$\begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta_0\exp[-\frac{1}{2}i(\varphi_0+\psi_0)] \\ -i\sin\frac{1}{2}\theta_0\exp[\frac{1}{2}i(\varphi_0-\psi_0)] \end{pmatrix}$$

by means of a rotation $R = \cos\frac{1}{2}\chi + i\boldsymbol{\sigma}\cdot(\mathbf{M}/M)\sin\frac{1}{2}\chi$, we arrive at

$$\cos\chi = \frac{M^2\cos\theta - M_z\bar{M}_{\bar{z}}}{[(M^2 - M_z^2)(M^2 - \bar{M}_{\bar{z}}^2)]^{\frac{1}{2}}},$$

that is,

$$\chi = 2MQ_2 = \arctan\frac{p_\theta\tan\theta}{M - (p_\varphi p_\psi/M\cos\theta)}.$$

physical systems provides an example of irreducible realizations for the rotation group. On the other hand, the irreducible realizations with a definite fixed value of $\mathfrak{I}$ are nevertheless very interesting since, as we shall see in forthcoming papers, they play a relevant role in the construction of the canonical realizations of the Galilei and Poincaré groups corresponding to a free particle with spin.

Such irreducible realizations have to be constructed following an axiomatic procedure.

We introduce two canonical variables q and p (having *a priori* no simple physical meaning) and we put [cf. Eqs. (31)]

$$M_x = (l^2 - p^2)^{\frac{1}{2}}\cos q, \quad M_y = (l^2 - p^2)^{\frac{1}{2}}\sin q, \quad M_z = p, \qquad (46)$$

where l^2 is a positive constant. Equations (15) are clearly satisfied by these expressions. Also, it holds $\mathfrak{I} \equiv M^2 = l^2$.

TABLE I. The variables of the typical form (Scheme B) for the realizations discussed in the present paper.

(1) Mass point

I	$P_1 = p_\varphi$	$Q_1 = \arctan \dfrac{\cos\varphi \cdot p_\theta - \sin\varphi \cot\theta \cdot p_\varphi}{-\sin\varphi \cdot p_\theta - \cos\varphi \cot\theta \cdot p_\varphi}$
II	$P_2 = p_\theta^2 + \dfrac{1}{\sin^2\theta} p_\varphi^2$	$Q_2 = \dfrac{1}{2P_2^{\frac{1}{2}}} \arctan \dfrac{p_\theta \tan\theta}{P_2^{\frac{1}{2}}}$
IV	$P_3 = p_r$	$Q_3 = r$

(2) Rotator

I	$P_1 = p_\varphi$	$Q_1 = \arctan \dfrac{\cos\varphi \cdot p_\theta - \sin\varphi \cot\theta \cdot p_\varphi}{-\sin\varphi \cdot p_\theta - \cos\varphi \cot\theta \cdot p_\varphi}$
II	$P_2 = p_\theta^2 + \dfrac{1}{\sin^2\theta} p_\varphi^2$	$Q_2 = \dfrac{1}{2P_2^{\frac{1}{2}}} \arctan \dfrac{p_\theta \tan\theta}{P_2^{\frac{1}{2}}}$

(3) Rigid body with fixed point[a]

I	$P_1 = p_\varphi$	$Q_1 = \arctan \dfrac{\sin\varphi \cdot p_\theta - \dfrac{\cos\varphi}{\sin\theta} p_\psi + \cos\varphi \cot\theta \cdot p_\varphi}{\cos\varphi \cdot p_\theta + \dfrac{\sin\varphi}{\sin\theta} p_\psi - \sin\varphi \cot\theta \cdot p_\varphi}$
II	$P_2 = p_\theta^2 + \dfrac{1}{\sin^2\theta}(p_\varphi^2 + p_\psi^2) - \dfrac{2\cot\theta}{\sin\theta} p_\varphi p_\psi$	$Q_2 = \dfrac{1}{2P_2^{\frac{1}{2}}} \arctan \dfrac{p_\theta \tan\theta}{P_2^{\frac{1}{2}} - \dfrac{p_\varphi p_\psi}{P_2^{\frac{1}{2}}}}$
IV	$P_3 = p_\psi$	$Q_3 = \arctan \dfrac{\sin\psi \cdot p_\theta - \dfrac{\cos\psi}{\sin\theta} p_\varphi + \cos\psi \cot\theta \cdot p_\psi}{\cos\psi \cdot p_\theta + \dfrac{\sin\psi}{\sin\theta} p_\varphi - \sin\psi \cot\theta \cdot p_\psi}$

[a] Note that the variables of the *typical form* in this case can also be viewed as defining the *Scheme B* for a particular class of realizations (the "symmetrical" ones) of the group $O(3) \times O(3) = O(4)$.

Now, the transformations corresponding to an infinitesimal rotation are given by [cf. Eqs. (32)]

$$\begin{aligned}\delta q &= \delta\omega_x \frac{p}{(l^2 - p^2)^{\frac{1}{2}}} \cos q \\ &\quad + \delta\omega_y \frac{p}{(l^2 - p^2)^{\frac{1}{2}}} \sin q - \delta\omega_z, \\ \delta p &= -\delta\omega_x (l^2 - p^2)^{\frac{1}{2}} \sin q + \delta\omega_y (l^2 - p^2)^{\frac{1}{2}} \cos q.\end{aligned} \tag{47}$$

The finite transformations can be constructed explicitly by integrating just the above differential relations. With this in view, we first integrate Eqs. (47) relative to rotations around the z and the x axes, respectively. Then we take advantage of the known property that a general rotation $R(\alpha, \beta, \gamma)$ characterized by the Euler angles α, β, γ can be expressed as a product of three rotations $R_z(\gamma) \cdot R_x(\beta) \cdot R_z(\alpha)$ of angles α, β, γ around the z, the x, and again the z axes. The consistency of the whole procedure is guaranteed by the fact that the system is integrable owing to the Lie theorems.

For a finite rotation ω_z around the z axis we have at once

$$\begin{aligned} q' &= q - \omega_z, \\ p' &= p. \end{aligned} \tag{48}$$

For a rotation around the x axis, the system (47) can be written

$$\begin{aligned} dq/d\omega_x &= [p/(l^2 - p^2)^{\frac{1}{2}}] \cos q, \\ dp/d\omega_x &= -(l^2 - p^2)^{\frac{1}{2}} \sin q. \end{aligned} \tag{49}$$

By eliminating the variable ω_x, we get

$$dp/dq = [(p^2 - l^2)/p] \tan q.$$

Then, integrating, it follows that

$$\cos q = C_1/(l^2 - p^2)^{\frac{1}{2}} \tag{50}$$

and, by inserting this expression into Eqs. (49), we obtain the general solution with a simple quadrature

$$\begin{aligned} q' &= \arctan\{[(l^2 - C_1^2)^{\frac{1}{2}}/C_1]\cos(\omega_x + C_2)\}, \\ p' &= -(l^2 - C_1^2)^{\frac{1}{2}}\sin(\omega_x + C_2), \end{aligned} \tag{51}$$

where C_1 and C_2 are integration constants. Finally, expressing C_1 and C_2 in terms of the initial values q and p, we have

$$\begin{aligned} q' = \arctan\{&\cos\omega_x \tan q \\ &+ \sin\omega_x[p/\cos q(l^2 - p^2)^{\frac{1}{2}}]\}, \\ p' = \cos&\,\omega_x p - \sin\omega_x \sin q(l^2 - p^2)^{\frac{1}{2}}. \end{aligned} \tag{52}$$

In conclusion, the general finite transformations we were looking for are deduced from Eqs. (48) and (52):

$$\begin{aligned} q' &= \arctan\left[\cos\beta\tan(q + \alpha) + \frac{\sin\beta\cdot p}{\cos(q+\alpha)\cdot(l^2 - p^2)^{\frac{1}{2}}}\right] - \gamma, \\ p' &= \cos\beta\cdot p - \sin\beta\sin(q+\alpha)\cdot(l^2 - p^2)^{\frac{1}{2}}. \end{aligned} \tag{53}$$

As a function in the group space, q is defined mod (2π). This is in agreement with the fact that, according to Eqs. (46), q is an angular coordinate, actually the angular specification of the projection of $\mathbf{M}$ in the plane (xy).

5. CONNECTIONS WITH SPINOR THEORY

In the preceding section, we constructed a faithful realization of the rotation group involving only two variables. Another (faithful) realization in two variables, but obviously noncanonical, can be obtained if we consider first the well-known transformation properties of the elementary spinor

$$\Phi = \begin{pmatrix}\eta \\ \xi\end{pmatrix}$$

and then if we form the ratio $\zeta = \eta/\xi$ between the two components (that is, if we consider η and ξ as homogeneous coordinates in a one-dimensional complex space). We presently see how the two realizations are directly related. Actually, the infinitesimal transformation properties of the elementary spinor can be written

$$\begin{aligned} \eta' &= \eta + \tfrac{1}{2}i[(\delta\omega_x - i\delta\omega_y)\xi + \delta\omega_z\eta], \\ \xi' &= \xi + \tfrac{1}{2}i[(\delta\omega_x + i\delta\omega_y)\eta - \delta\omega_z\xi], \end{aligned} \tag{54}$$

from which it follows that

$$\zeta' = \zeta + \tfrac{1}{2}i\delta\omega_x(1 - \zeta^2) + \tfrac{1}{2}\delta\omega_y(1 + \zeta^2) + i\delta\omega_z\zeta. \tag{55}$$

Now, we try to connect in a complete general way the spinor

$$\begin{pmatrix}\eta \\ \xi\end{pmatrix}$$

with the vector $\mathbf{M}$ without any reference, for the moment, to a particular realization. To this aim, we identify $\mathbf{M}$ with the real vector which can be associated to any elementary spinor.[6-8] We have

$$\begin{aligned} M_x &= \xi\eta^* + \xi^*\eta, \\ M_y &= -i(\xi\eta^* - \xi^*\eta), \\ M_z &= \eta\eta^* - \xi\xi^*, \\ M &= \eta\eta^* + \xi\xi^*. \end{aligned} \tag{56}$$

Then, from Eq. (18), we deduce

$$\begin{aligned} \mathfrak{Q} &= \arg\xi - \arg\eta, \\ \mathfrak{P} &= \eta\eta^* - \xi\xi^*, \end{aligned} \tag{57}$$

and solving for η and ξ we can write

$$\begin{aligned} \eta &= [\tfrac{1}{2}(M + \mathfrak{P})]^{\frac{1}{2}} e^{-\frac{1}{2}i\mathfrak{Q}+i\alpha}, \\ \xi &= [\tfrac{1}{2}(M - \mathfrak{P})]^{\frac{1}{2}} e^{+\frac{1}{2}i\mathfrak{Q}+i\alpha}, \end{aligned} \tag{58}$$

where α is an arbitrary function of all the variables on which the realization operates. Finally, we get

$$\zeta = [(M + \mathfrak{P})/(M - \mathfrak{P})]^{\frac{1}{2}} e^{-i\mathfrak{Q}}. \tag{59}$$

Now, from Eq. (59) it follows that

$$\begin{aligned} \{M_x, \zeta\} &= \tfrac{1}{2}i(1 - \zeta^2), \\ \{M_y, \zeta\} &= \tfrac{1}{2}(1 + \zeta^2), \quad \{M_z, \zeta\} = i\zeta, \end{aligned} \tag{60}$$

so that, as it could be expected, the transformation properties of ζ as they are deduced from Eq. (59) agree with Eq. (55). If we consider again the irreducible realization, we find the connection we were looking for by simply setting, according to Eq. (59),

$$\zeta = [(l + p)/(l - p)]^{\frac{1}{2}} e^{-iq} \tag{61}$$

or inversely,

$$\begin{aligned} q &= -\arg\zeta, \\ p &= l[(|\zeta|^2 - 1)/(|\zeta|^2 + 1)]. \end{aligned} \tag{62}$$

It can be easily proved that no function α of only $\mathfrak{Q}$, $\mathfrak{P}$ (and $\mathfrak{I}$) in Eqs. (58) can reproduce Eqs. (54), so that it is not possible in any way to connect the irreducible realizations with the transformation properties of the spinor components themselves. This is quite obvious

[6] See, for instance, Refs. (7) and (8). Our association is slightly different from those given there.

[7] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1931), pp. 144–45.

[8] W. T. Payne, Am. J. Phys. **20**, 253 (1952).

if one notices that of the four quantities which characterize the spinor, one is left unchanged by the rotations while the other three actually transform. This remark suggests also that the only realization of the rotation group which could be connected with the transformation properties of the elementary spinor is the realization which, in its *typical form*, contains a variable conjugate to the invariant $\mathfrak{J}$, that is, essentially, *the canonical realization corresponding to the rotator.* As a matter of fact, such a connection can be expressed by putting

$$\begin{aligned}\eta &= [\tfrac{1}{2}(P_2^{\frac{1}{2}} + P_1)]^{\frac{1}{2}} \exp(-\tfrac{1}{2}iQ_1 - iP_2^{\frac{1}{2}}Q_2),\\ \xi &= [\tfrac{1}{2}(P_2^{\frac{1}{2}} - P_1)]^{\frac{1}{2}} \exp(+\tfrac{1}{2}iQ_1 - iP_2^{\frac{1}{2}}Q_2).\end{aligned} \tag{63}$$

This connection is directly related to the geometrical interpretation of the spinor if the angles $2P_2^{\frac{1}{2}}Q_2$ (cf. Sec. 3ii), Q_1 and arcos $P_1/P_2^{\frac{1}{2}}$ are identified with the angles $-\psi$, $-\varphi$, and θ of Ref. 8, respectively.

Let us emphasize, finally, that the factor $\frac{1}{2}$ appearing in front of the angles Q_1 and $2P_2^{\frac{1}{2}}Q_2$ in Eqs. (63) corresponds to the fact that the transformation properties of the spinor, *unlike the canonical realization*, provide a realization of the universal covering group rather than of the rotation group itself.

APPENDIX A. INTEGRATION OF EQ. (28) AND DETERMINATION OF THE VARIABLE Q_2 FOR THE MASS POINT

The equation to be integrated is

$$(p_\theta^2 \sin^2\theta + p_\varphi^2\cos^2\theta)\frac{\partial g}{\partial p_\varphi} - p_\varphi p_\theta \frac{\partial g}{\partial p_\theta} + p_\varphi \sin\theta\cos\theta\frac{\partial g}{\partial \theta} = 0. \tag{A1}$$

It is easy to verify the obvious result that a solution of Eq. (A1) is

$$g_1 \equiv M^2 = p_\theta^2 + (1/\sin^2\theta)p_\varphi^2.$$

Thus, we are interested in a particular solution independent of g_1. We search for a solution of the form

$$g(\theta, p_\theta) = U(\theta)\cdot V(p_\theta). \tag{A2}$$

It follows that

$$-p_\theta\frac{1}{V}\frac{dV}{dp_\theta} + \sin\theta\cos\theta\frac{1}{U}\frac{dU}{d\theta} = 0,$$

that is,

$$p_\theta\frac{1}{V}\frac{dV}{dp_\theta} = k, \qquad \sin\theta\cos\theta\frac{1}{U}\frac{dU}{d\theta} = k. \tag{A3}$$

Solving these equations, we find

$$V = C_1 p_\theta^k \tag{A4}$$

and

$$\begin{aligned}\log U &= k\int\frac{2\,d\theta}{\sin 2\theta} = -\frac{k}{2}\log\frac{1+\cos 2\theta}{1-\cos 2\theta} + \log C_2\\ &= \log\tan^k\theta + \log C_2,\end{aligned}$$

that is,

$$U = C_2\tan^k\theta \tag{A5}$$

(C_1 and C_2 integration constants). Finally, by choosing $C_1 = C_2 = k = 1$, we obtain

$$g_2 = p_\theta\tan\theta. \tag{A6}$$

Any other solution of Eq. (A1) must be a function of g_1 and g_2. Then, let us consider a function $\Psi(g_1, g_2)$ and impose $\{\Psi, M^2\} \equiv \{g_2, M^2\}(\partial\Psi/\partial g_2) = 1$. Since $\{g_2, M^2\} = 2g_2^2 + 2g_1^2$, we conclude

$$\Psi = \int\frac{dg_2}{2g_2^2 + 2g_1^2} = \frac{1}{2M}\arctan\frac{p_\theta\tan\theta}{M} + \text{const.} \tag{A7}$$

APPENDIX B. THE FINITE TRANSFORMATIONS OF THE VARIABLE Q_2

Owing to what was said in Sec. 4, it is enough to consider rotations around the x axis. Thus, we have

$$Q_2' = Q_2 - \delta\omega_x[\cos Q_1/2(P_2 - P_1^2)^{\frac{1}{2}}]. \tag{B1}$$

From the solutions of the first two Eqs. (32) [cf. Sec. 4, Eqs. (47)–(51); cf. also Eqs. (33)], we obtain

$$\frac{dQ_2}{d\omega_x} = -\frac{C_1}{2}\frac{1}{P_2 - (P_2 - C_1^2)\sin^2(\omega_x + C_2)} \tag{B2}$$

(C_1 and C_2 integration constants). Finally, with a quadrature, it follows that

$$\begin{aligned}Q_2' &= Q_2 - \frac{1}{2P_2^{\frac{1}{2}}}\arctan\left[\frac{C_1}{P_2^{\frac{1}{2}}}\tan(\omega_x + C_2)\right]\\ &= Q_2 - \frac{1}{2P_2^{\frac{1}{2}}}\arctan\left[\frac{\tan\beta\sin Q_1\cos Q_1\cdot(P_2 - P_1^2) - \cos Q_1\cdot P_1(P_2 - P_1^2)^{\frac{1}{2}}}{\tan\beta\cdot P_1P_2 + \sin Q_1\cdot P_2(P_2 - P_1^2)^{\frac{1}{2}}}\right].\end{aligned} \tag{B3}$$

APPENDIX C. THE TRANSFORMATIONS OF THE FIRST PARAMETER GROUP

The body configuration can be expressed in terms of the matrix $S_{\varphi,\theta,\psi}$ connecting the coordinates in the space system to the coordinates in the intrinsic system. The transformation properties of the Euler angles φ, θ, ψ under a rotation $S_{\alpha,\beta,\gamma}$ can be implicitly written as

$$S^{-1}_{\varphi',\theta',\psi'} = S_{\alpha,\beta,\gamma} \cdot S^{-1}_{\varphi,\theta,\psi}. \tag{C1}$$

In the spinor representation, the above relation becomes

$$\begin{aligned}&\begin{pmatrix} e^{-\frac{1}{2}i(\varphi'+\psi')}\cos\frac{1}{2}\theta' & -ie^{-\frac{1}{2}i(\varphi'-\psi')}\sin\frac{1}{2}\theta' \\ -ie^{\frac{1}{2}i(\varphi'-\psi')}\sin\frac{1}{2}\theta' & e^{\frac{1}{2}i(\varphi'+\psi')}\cos\frac{1}{2}\theta' \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{1}{2}i(\alpha+\gamma)}\cos\frac{1}{2}\beta & ie^{-\frac{1}{2}i(\alpha-\gamma)}\sin\frac{1}{2}\beta \\ ie^{\frac{1}{2}i(\alpha-\gamma)}\sin\frac{1}{2}\beta & e^{-\frac{1}{2}i(\alpha+\gamma)}\cos\frac{1}{2}\beta \end{pmatrix} \\ &\times \begin{pmatrix} e^{-\frac{1}{2}i(\varphi+\psi)}\cos\frac{1}{2}\theta & -ie^{-\frac{1}{2}i(\varphi-\psi)}\sin\frac{1}{2}\theta \\ -ie^{\frac{1}{2}i(\varphi-\psi)}\sin\frac{1}{2}\theta & e^{\frac{1}{2}i(\varphi+\psi)}\cos\frac{1}{2}\theta \end{pmatrix}.\end{aligned} \tag{C2}$$

More simply, we can characterize the configuration of the body by using directly the spinor

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{-\frac{1}{2}i(\varphi+\psi)}\cos\frac{1}{2}\theta \\ -ie^{\frac{1}{2}i(\varphi-\psi)}\sin\frac{1}{2}\theta \end{pmatrix}$$

formed by the first column of the matrix $S^{-1}_{\varphi,\theta,\psi}$. Actually

$$\begin{aligned}&\begin{pmatrix} e^{-\frac{1}{2}i(\varphi'+\psi')}\cos\frac{1}{2}\theta' \\ -ie^{\frac{1}{2}i(\varphi'-\psi')}\sin\frac{1}{2}\theta' \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{1}{2}i(\alpha+\gamma)}\cos\frac{1}{2}\beta & ie^{-\frac{1}{2}i(\alpha-\gamma)}\sin\frac{1}{2}\beta \\ ie^{\frac{1}{2}i(\alpha-\gamma)}\sin\frac{1}{2}\beta & e^{-\frac{1}{2}i(\alpha+\gamma)}\cos\frac{1}{2}\beta \end{pmatrix} \\ &\times \begin{pmatrix} e^{-\frac{1}{2}i(\varphi+\psi)}\cos\frac{1}{2}\theta \\ -ie^{\frac{1}{2}i(\varphi-\psi)}\sin\frac{1}{2}\theta \end{pmatrix},\end{aligned} \tag{C3}$$

(cf. Ref. 8. The conventions adopted in this paper are different from ours.) Then, solving Eqs. (C3) with respect to φ', θ', ψ', the relations (36) are deduced.

Alternatively, in terms of the rotation angle ω about the axis $\hat{n}$, we may write

$$\begin{aligned}\begin{pmatrix} e^{-\frac{1}{2}i(\varphi'+\psi')}\cos\frac{1}{2}\theta' \\ -ie^{\frac{1}{2}i(\varphi'-\psi')}\sin\frac{1}{2}\theta' \end{pmatrix} &= [\cos\tfrac{1}{2}\omega + i\boldsymbol{\sigma}\cdot\hat{n}\sin\tfrac{1}{2}\omega] \\ &\times \begin{pmatrix} e^{-\frac{1}{2}i(\varphi+\psi)}\cos\frac{1}{2}\theta \\ -ie^{\frac{1}{2}i(\varphi-\psi)}\sin\frac{1}{2}\theta \end{pmatrix},\end{aligned} \tag{C4}$$

from which, for instance, the infinitesimal transformations (37) can be obtained.

APPENDIX D. INTEGRATION OF THE SYSTEM (42) AND DETERMINATION OF THE VARIABLE Q_2 FOR THE RIGID BODY

The system to be integrated is

$$\begin{aligned}&\left[p_\theta^2 + \left(\frac{p_\psi}{\sin\theta} - \cot\theta\cdot p_\varphi\right)^2\right]\frac{\partial g}{\partial p_\varphi} \\ &\quad + \left[\frac{p_\theta}{\sin\theta}\left(\cot\theta\cdot p_\psi - \frac{p_\varphi}{\sin\theta}\right)\right]\frac{\partial g}{\partial p_\theta} \\ &\quad - \left[\frac{p_\psi}{\sin\theta} - \cot\theta\cdot p_\varphi\right]\frac{\partial g}{\partial\theta} = 0, \\ &\left[p_\theta^2 + \left(\frac{p_\varphi}{\sin\theta} - \cot\theta\cdot p_\psi\right)^2\right]\frac{\partial g}{\partial p_\psi} \\ &\quad + \left[\frac{p_\theta}{\sin\theta}\left(\cot\theta\cdot p_\varphi - \frac{p_\psi}{\sin\theta}\right)\right]\frac{\partial g}{\partial p_\theta} \\ &\quad - \left[\frac{p_\varphi}{\sin\theta} - \cot\theta\cdot p_\psi\right]\frac{\partial g}{\partial\theta} = 0.\end{aligned} \tag{D1}$$

A solution is obviously provided by $\bar{g}_1 \equiv M^2$ [cf. Eq. (43)]. We have to look for a second solution $\bar{g}_2$ independent of $\bar{g}_1$. Using as independent variables M^2, p_φ, p_ψ and $\xi \equiv g_2 = p_\theta \tan\theta$, it follows that

$$\begin{aligned}\{Q_1, g\} &\equiv \frac{\partial g}{\partial p_\varphi} - \frac{\xi p_\psi}{(M^2 - p_\varphi^2)\cos\theta}\frac{g}{\partial\xi} = 0, \\ \{Q_3, g\} &\equiv \frac{\partial g}{\partial p_\psi} - \frac{\xi p_\varphi}{(M^2 - p_\psi^2)\cos\theta}\frac{\partial g}{\partial\xi} = 0,\end{aligned} \tag{D2}$$

where $\cos\theta$ still has to be re-expressed in terms of $M^2, p_\varphi, p_\psi, \xi$. Multiplying by p_φ and p_ψ, respectively, and subtracting, we get

$$\frac{\partial g}{\partial p_\varphi} = \frac{p_\psi(M^2 - p_\psi^2)}{p_\varphi(M^2 - p_\varphi^2)}\frac{\partial g}{\partial p_\psi}. \tag{D3}$$

To solve Eq. (D3) we use the method of the characteristics (see for instance Ref. 9). We can write

$$dp_\psi/dp_\varphi = -p_\psi(M^2 - p_\psi^2)/p_\varphi(M^2 - p_\varphi^2), \tag{D4}$$

from which

$$\log\frac{p_\psi^2}{M^2 - p_\psi^2} = -\log\frac{p_\varphi^2}{M^2 - p_\varphi^2} + \log\Omega. \tag{D5}$$

The integration constant

$$\Omega = [p_\psi^2/(M^2 - p_\psi^2)]\cdot[p_\varphi^2/(M^2 - p_\varphi^2)] \tag{D6}$$

[9] E. Goursat, *A Course in Mathematical Analysis* (Dover Publications, Inc., New York, 1959), Vol. II, Part Two, *Differential Equations*, Chap. II, Sec. 3. For a more detailed discussion, see for instance, R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1962), Vol. II, "Partial Differential Equations," Chap. I, Sec. 5.

is a solution of Eq. (D3). The same is true for a function of Ω, M^2, ξ. Now, using such a kind of function, for instance in the first of the Eqs. (D2), we obtain

$$\Omega \frac{\partial g}{\partial \Omega} - \frac{\xi}{2M^2} \frac{p_\varphi p_\psi}{\cos\theta} \frac{\partial g}{\partial \xi} = 0, \qquad \text{(D7)}$$

and observing that, owing to Eq. (40), $\cos\theta$ is expressed in terms of the independent variables in the form

$$\cos\theta = \frac{p_\varphi p_\psi}{\xi^2 + M^2}\left\{1 \pm \left[1 + \left(1 + \frac{\xi^2}{M^2}\right)\left(\frac{1}{\Omega} - 1\right)\right]^{\frac{1}{2}}\right\}, \qquad \text{(D8)}$$

we get

$$\Omega \frac{\partial g}{\partial \Omega} - \frac{\xi(\xi^2 + M^2)}{2M^2} \times \frac{1}{1 \pm \{1 + [1 + (\xi^2/M^2)][(1/\Omega) - 1]\}^{\frac{1}{2}}} \frac{\partial g}{\partial \xi} = 0. \qquad \text{(D9)}$$

Finally, performing the following change of independent variables

$$(1/\Omega) - 1 = t,$$
$$\pm\{1 + [1 + (\xi^2/M^2)][(1/\Omega) - 1]\}^{\frac{1}{2}} = z + 1, \qquad \text{(D10)}$$

we arrive at the form

$$2t(t+1)(\partial g/\partial t) + z(z + t + 2)(\partial g/\partial z) = 0. \qquad \text{(D11)}$$

Then, if we apply again the method used above, we obtain the equation

$$\frac{dz}{dt} = \frac{1}{2t(t+1)} z^2 + \frac{t+2}{2t(t+1)} z, \qquad \text{(D12)}$$

which is of a Bernoulli type. Standard methods give now the solution

$$z = t/[1 + C(t+1)^{\frac{1}{2}}], \qquad \text{(D13)}$$

where C is the constant of integration. Again, the expression

$$C = [1/(t+1)^{\frac{1}{2}}][(t/z) - 1] \qquad \text{(D14)}$$

is a solution of Eq. (D11) and, consequently, of the system (D2). Thus, going back to the original variables we obtain the solution

$$\bar{g}_2(M^2, \theta, p_\varphi, p_\psi) = \frac{M^2 \cos\theta - p_\varphi p_\psi}{[(M^2 - p_\varphi^2)(M^2 - p_\psi^2)]^{\frac{1}{2}}}. \qquad \text{(D15)}$$

Finally, since $\{\bar{g}_2, M^2\} = -2M(1 - \bar{g}_2^2)^{\frac{1}{2}}$ we have

$$\{(2M)^{-1} \operatorname{arcos} \bar{g}_2, M^2\} = 1 \qquad \text{(D16)}$$

and we can conclude

$$Q_2 = \frac{1}{2M} \operatorname{arcos} \bar{g}_2 = \frac{1}{2M} \arctan \frac{p_\theta \tan\theta}{M - (p_\varphi p_\psi / M\cos\theta)}. \qquad \text{(D17)}$$

Connection between the Marchenko Formalism and N/D Equations: Regular Interactions. I*

HENRI CORNILLE
Institut de Physique Nucléaire, Division de Physique Théorique, 91 Orsay, France

(Received 14 September 1966)

In this paper and in the following, we study, in potential scattering, the existence and meaning of the solutions of the N/D equations in the equivalent formulation f/f. For S waves, considering only regular discontinuity $\mu\Delta(x)$, such that the resulting integral equation is of the Fredholm type, we study the corresponding Fredholm determinant $\mathfrak{D}(\mu)$. We remark that Marchenko formalism gives exactly the same resulting equation and then we have the possibility to interpret in terms of local potentials. We show the connection between the nth trace of the kernel of the resulting integral equation and the nth term of the potential reconstructed from the discontinuity. The connection between dispersion relation and the corresponding potential reconstructed from the discontinuity is given by the relation

$$\mathfrak{D}(\mu) = \exp -\frac{1}{2}\int_0^\infty \int_r^\infty V(t, \mu)\, dt\, dr.$$

In the present paper we limit our study to $|\mu|$ less than the smallest modulus root of $\mathfrak{D}(\mu)$ where a perturbation expansion of the solution exists and we show that $V(r, \mu)$ is regular at $r = 0$. On the other hand, for Yukawa-type potentials where the inverse Laplace transform is $\lambda C(\alpha)$ the Fredholm determinant is $\exp\,(-\int_m^\infty \lambda C(\alpha)/\alpha^2\, d\alpha)$ and cannot vanish such that the corresponding solutions of the resulting integral equation exist always.

I. INTRODUCTION

IN dispersion theoretic calculation, during the recent years, the N/D equations[1] have been extensively used as dynamical equations for strong-interaction physics. In order to understand more carefully the meaning of the approach, many works have been made in potential scattering,[2,3] mainly by using the discontinuity given by the Born amplitude. In this paper we are interested for "regular interactions" in the alternative approach f/f, where f is the Jost function.[4,5] Although this can seem at first sight strange, it appears that physicists in general have not been interested, even in the simplified version given by potential scattering, in the problem of the existence and uniqueness of these solutions. In fact the resulting integral equation of these f/f (or N/D) equations has a kernel proportional to the whole discontinuity $\Delta(x)$ of the S matrix. Our aim is the following: we consider, as usual, the discontinuity Δ as input in the resulting integral equation; from the corresponding solutions, we can reconstruct the Jost function, the S matrix, the physical states, and physical quantities. We study the possibility of such a reconstruction and try to understand the reason of possible breakdown appearing in this way. For "regular interaction" we assume

$$\int_{m/2}^\infty \left|\frac{\Delta(x)}{x}\right| dx < \infty$$

and that the resulting integral equation is of the Fredholm type and in order to investigate the existence and uniqueness of the solution we have to seek the roots of the Fredholm determinant corresponding to the kernel $\Delta(y)/(x + y)$.

At this stage we emphasize that we can adopt two entirely different points of view.

First, in order to keep the character of linear operator for the resulting integral equation, we put formally the discontinuity equal to $\mu\Delta(x)$ (μ is a parameter). Then we have still a "regular interaction." In this case the corresponding potential $V(r, \mu)$ is not linear in μ and does not depend in a trivial manner on the parameter μ;

$$V(r, \mu) = \sum_{n=1}^\infty \mu^n V_n(r),$$

where $V_n(r)$ is determined by $\Delta(x)$.

Second, we can consider that the discontinuity is $\Delta(x, \lambda)$, where Δ is a complicated function of λ, nonlinear in λ. In this case λ is the parameter such that

$$\Delta(x, \lambda) = \sum_{n=1}^\infty \lambda^n \Delta_n(x),$$

* Paper presented at the 1966 International conference on High Energy Physics at Berkeley, Calif,

[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] R. Blankenbecler, H. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N.Y.) **10**, 62 (1960); J. D. Bjorken and A. Goldberg, Nuovo Cimento **16**, 539 (1960); M. Luming, Phys. Rev. **136**, B1120 (1964); B. Kantor, Ann. Phys. (N.Y.) **33**, 196 (1965); S. M. Roy and R. Blankenbecler, *ibid*, **35**, 314 (1965).

[3] E. M. Nyman, Nuovo Cimento **37**, 429 (1965).

[4] A. Martin, Nuovo Cimento **19**, 1257 (1961); Nuovo Cimento Suppl. **21**, 157 (1961); in *Progress in Elementary Particles and Cosmic Ray Physics* (North-Holland Publishing Company, Amsterdam, 1963).

[5] V. de Alfaro and T. Regge, Nuovo Cimento **20**, 956 (1961).

where $\Delta_n(x)$ is the discontinuity coming from the nth Born approximation. In this case λ is a linear parameter for the potential $\lambda V(r)$ and $\Delta_n(x)$ is determined from $V(r)$. We know from Martin[4] results that for Yukawa type of potentials $\Delta(x, \lambda)/x$ is integrable when $x \to \infty$; then for these interactions our above conditions for "regular interactions" are satisfied.

In the first case the resulting integral equation is linear in μ, in the second case, the Schrödinger equation is linear in λ. We emphasize that the families of interactions we can obtain when λ or μ goes from $-\infty$ to $+\infty$ are different.

When we adopt the first point of view we have to investigate the roots of the Fredholm determinant $\mathfrak{D}(\mu)$. De Alfaro and Regge[6] have given a sufficient condition

$$\int_{m/2}^{\infty} \left| \frac{\mu\Delta(x)}{x} \right| dx < 2 \log 2$$

such that $\mathfrak{D}(\mu)$ cannot vanish. But this result does not recover the whole Yukawa-type family of potentials such that even if it has been proved that the Yukawa-type family leads to this resulting integral equation, it has not yet been proved inversely starting from this resulting integral equation (although this is plausibly true) the existence and uniqueness of the solutions corresponding to the whole Yukawa family. From the mathematical point of view, the resulting integral equation has a polar nondegenerate kernel and we know that in general there exists an infinity of singular values $\mu_{\pm j}$ such that $\mathfrak{D}(\mu_{\pm j}) = 0$ ($\mu_j > 0$, $\mu_{-j} < 0$).

But $\mathfrak{D}(\mu)$ is an entire function of μ with coefficients depending in a nontrivial manner on $\Delta(x)$; then it does not appear very easy to find the localization as well as the meaning of these $\mu_{\pm j}$. Because at first sight the mathematical point of view does not seem to be of great help, perhaps in order to have some insight, it will be better to try to understand more carefully what we seek physically. In fact what we have to solve is nothing else but the Jost function for families of potentials nonlinear in the coupling parameter and we know that for "honest" regular interactions (like the Yukawa-type family) there exists for each value of the coupling constant λ of the potential, a unique solution. We can formulate the problem in another way. It is very usual to invoke some physical requirements in order to understand singularities and breakdown in a theory (bound, states, resonances, spins). From this point of view it seems difficult in the case of physically acceptable, regular interaction, to attribute some physical meaning to the nonexistence of the Jost function for some special μ values. Then we can perhaps make the conjecture that although the mathematical requirements are such that there exists an infinity of singular $\mu_{\pm j}$ values, the physical conditions are such that we never attain these values for physically acceptable regular interactions like those of the Yukawa-type family. It is the aim of this paper and of the following to verify that these physical requirements are satisfied mathematically. In fact from Martin's work[4] and the De Alfaro and Regge condition,[5] we see that $|\mu|$ small enough will include at least a part of the Yukawa-type family. But we have the feeling that we can go outside this interval given by the De Alfaro and Regge condition; perhaps up to the first positive and the first negative singular values $\mu_{\pm 1}$ [smallest modulus >0 or <0 root of $\mathfrak{D}(\mu)$], where we have also the desire to find Yukawa-type interactions or some generalization of Yukawa-type interactions.

For μ equal or outside these values where physical insight cannot help us, we feel that we will find interactions violating strongly some conditions of the Yukawa family or physically nonacceptable states like ghosts. (We do not consider CDD poles in these papers.)

The answer to all these conjectures will be given by the powerful Marchenko[6] inversion formalism. Note that our problem is similar to an inverse scattering problem. We give as scattering data the discontinuity which is our input and try to interpret the corresponding interactions in terms of potentials. In order to avoid ambiguities, first we adopt the point of view that the physical states or bound states are given by the interaction; second we do not consider the problem of Bargman phase-equivalent potentials. We must add that Marchenko equations have been established with the restrictions of the existence of the moments for the potentials but we do not retain these conditions for the following reason: We remark that Marchenko formalism gives exactly the same equations as the resulting integral equation coming from f/f. Then the problem of the existence and meaning of the solutions in the two formalisms are the same if we give the same input $\mu\Delta(x)$, but we have the advantage in Marchenko formalism of having a direct interpretation in terms of potentials and consequently in terms of bound-state wavefunctions. Our fundamental result from Marchenko formalism is

$$V(\mu, r) = -2 \frac{d}{dr} K(r, r); \quad K(r, r) = \frac{(d/dr)\mathfrak{D}(\mu, r)}{\mathfrak{D}(\mu, r)},$$

where $\mathfrak{D}(\mu, r)$ is the Fredholm denominator of the Jost solution and $\mathfrak{D}(\mu, 0) = \mathfrak{D}(\mu)$.

[6] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory* (Gordon and Breach Science Publishers, Inc., New York, 1963).

We get

$$\mathfrak{D}(\mu) = \exp\left(-\frac{1}{2}\int_0^\infty\int_r^\infty V(\mu, t)\,dt\,dt\right),$$

$$\mathfrak{D}(\mu) = \exp\left(-\int_{m/2}^\infty \frac{C(2x, \mu)}{(2x)^2}\,dx\right),$$

or

$$\mathfrak{D}(\mu) = \exp\left(-\int_{m/2}^\infty \frac{B(x, \mu)}{2x}\,dx\right),$$

where C is the inverse Laplace transform of the potential and B the discontinuity coming from the first Born approximation.

If we adopt the second point of view where the discontinuity is $\Delta(x, \lambda)$ (see above), then our fundamental result means that the Fredholm determinant of the resulting integral equation is

$$\exp\left(-\int_m^\infty \lambda\frac{C(\alpha)}{2\alpha^2}\,d\alpha\right),$$

showing for instance that a Fredholm solution exists always for Yukawa-type potentials.

We find that the general expression of the potential reconstructed from the discontinuity $\mu\Delta(x)$ is

$$V(\mu, r) = 2\sum_{n=1}^\infty \frac{\mu^n}{n}\int_{m/2}^\infty dx_1\cdots\int_{m/2}^\infty dx_n \times \frac{\left(\prod_{i=1}^n \Delta(x_i)e^{-2rx_i}\right)\left(\sum_{j=1}^n 2x_j\right)^2}{(x_1+x_2)(x_2+x_3)\cdots(x_{n-1}+x_n)(x_n+x_1)}.$$

We have also shown the connection between the nth trace A_n of the resulting integral equation and the nth term of the inverse Laplace transform of the potential, reconstructed from the discontinuity following Martin's method[4]: We get

$$\frac{A_n}{n} = \int_m^\infty \frac{C_n(\alpha)}{2\alpha^2}\,d\alpha.$$

Then for $|\mu|$ less than the smallest modulus root of $\mathfrak{D}(\mu)$, both the series

$$\sum \mu^n \frac{A_n}{n} \quad\text{and}\quad \sum \mu^n \int_m^\infty \frac{C_n(\alpha)}{2\alpha^2}\,d\alpha$$

are convergent and we show that the corresponding $V(\mu, r)$ is "regular" at the origin. In the following paper the study of $|\mu|$ outside this smallest modulus root will be made.

II. f/f EQUATIONS AND THE RESULTING INTEGRAL EQUATION

We want to study the existence and meaning of the solutions of the so-called N/D equations in potential scattering for "regular interactions." Here we limit ourselves to S waves. As is well known, another equivalent approach is the f/f formalism, where $f(k)$ is the Jost function such that $f(-k) = D(k^2)$. For simplicity we consider the second approach because in this case we have only to investigate one resulting integral equation. We recall briefly the results[4] of this approach.

We assume that the potential is of the Yukawa type[7]

$$V(r) = \int_m^\infty e^{-\alpha r}C(\alpha)\,d\alpha, \tag{1a}$$

$V(r)$ is holomorphic for

$$\operatorname{Re} r > 0, \tag{1b}$$

$$\int_m^\infty \left|\frac{C(\alpha)}{\alpha^2}\right| d\alpha < \infty. \tag{1c}$$

Equation (1a) means that $V(r)$ is a Laplace transform, mainly there exists a half-plane $\operatorname{Re} r > c$ where $V(r)$ is holomorphic; Eq. (1b) means that $c \le 0$ and avoids, for instance, poles for $V(r)$ in $\operatorname{Re} r > 0$; and Eq. (1c) means mainly that V is "regular" near the origin (less singular than $r^{-\eta}$, $\eta < 2$).

We consider the Jost solution

$$f(k, r) \underset{r\to 0}{\simeq} e^{-ikr}$$

of the Schrödinger equation and we define as usual the Jost function

$$f(k) = \lim_{r\to 0} f(k, r).$$

For the family of equations (1), studying the analytical properties of $f(k)$, shows that $f(k)$ is analytic in the k complex plane, with a cut beginning at $\frac{1}{2}im$ along the positive imaginary k axis. Furthermore, outside the cut,

$$|f(k) - 1| \xrightarrow[|k|\to\infty]{} 0$$

sufficiently rapidly such that the following spectral representation can be obtained[4]:

$$f(k) = 1 - i\int_{m/2}^\infty \frac{R(y)}{k - iy}\,dy,$$

where we have used the property $f^*(-k^*) = f(k)$ and where $R(y)$ is real. Using this representation for the S matrix, $S(k) = f(k)/f(-k)$; then $R(y)$ can be obtained from the discontinuity of the S matrix along the cut $[\frac{1}{2}im, i\infty]$. We find $R(y) = \Delta(y)f(-iy)$, where Δ is minus the discontinuity of S. (For simplicity, we have changed the sign of the discontinuity throughout this paper.) Then we get the following integral representation for the Jost function

$$f(k) = 1 - i\int_{m/2}^\infty \frac{f(-ix)\Delta(x)\,dx}{k - ix}, \tag{2}$$

$$2i\pi\Delta(x) = -[S(ix + \epsilon) - S(ix - \epsilon)], \quad x > \tfrac{1}{2}m, \tag{3}$$

[7] See for instance V. de Alfaro and T. Regge, *Potential Scattering* (North-Holland Publishing Company, Amsterdam, 1965), p. 48.

and the problem is reduced[4] to solving the resulting integral equation

$$F(x) = 1 + \int_{m/2}^{\infty} \frac{F(y)\Delta(y)\,dy}{x+y}, \qquad F(x) = f(-ix), \tag{4}$$

which because of Eq. (1c), $\Delta(y)/y$ is integrable[4] when y goes to ∞ and (4) is of the Fredholm type for the family of equation (1).

In fact the existence of solutions of Eq. (4) has been proved by De Alfaro and Regge[5] only for

$$\int_{m/2}^{\infty} \left| \frac{\Delta(x)}{x} \right| dx < 2 \log 2. \tag{5}$$

We note that this sufficient condition (5) does not recover all of the family of equations (1). For instance if $C(\alpha)$ is <0, then[4] Δ is <0 and Eq. (5) can be violated, but in this case the Fredholm determinant of Eq. (4) is always different of zero and we always have a solution.

So is the way going from potential scattering to dispersion relations in this formulation f/f. We are interested in this paper and in the following in the inverse problem. Our starting point is the integral equation (4) where we write formally the discontinuity as $\mu\Delta(x)$ (μ is a parameter, see the Introduction).

Then

$$F(x) = 1 + \mu \int_{m/2}^{\infty} \frac{\Delta(y)F(y)}{x+y}\,dy. \tag{6}$$

We assume a "regular interaction" for $\Delta(x)$ real such that

$$\int_{m/2}^{\infty} \left| \frac{\Delta(x)}{x} \right| dx < \infty$$

and

$$\int_{m/2}^{\infty}\int_{m/2}^{\infty} \left(\frac{\Delta(y)}{x+y}\right)^2 dx\,dy < \infty.^{8} \tag{7}$$

Then (6) is of the Fredholm type.

We seek the existence of solutions of (6) with the conditions (7) and will try to interpret[9] in terms of local potentials of the type (1a). Then we have the following dilemma:

On the one hand at least for the family of equations (1) we know that $F(x) = f(-ix)$, $(x > 0)$ being the Jost function exists always for "honest interactions" and can be constructed for instance from Volterra-type integral equations of the Schrödinger equation corresponding to Jost solutions. On the other hand from the mathematical point of view Eqs. (6) and (7) are Fredholm integral equations with polar[10] non-degenerate kernel and we know that there exist in general singular values $\mu_{\pm j}$ such that the Fredholm determinant $\mathfrak{D}(\mu)$ of (6) vanishes: $\mathfrak{D}(\mu_{\pm j}) = 0$. For instance, for the previously considered case $C < 0$, $\Delta < 0$, (6) can be reduced to a Hilbert–Schmidt symmetric real kernel with an infinite number of singular real values $\mu_{-j} < 0$. Then, in order to reconcile this apparent contradiction, we have the feeling that even if we can show for family (1) that $\mathfrak{D}(\mu)$ never vanishes, in the inverse problem we certainly find (at least in order to explain these singular values $\mu_{\pm j}$) other potentials for which Eqs. (1b) or (1c) or both will be rejected. In other words, if from Eqs. (1), Eq. (4) was obtained, then certainly from (6) and (7) more than Eqs. (1) will be obtained.

In fact our aim will be the study of the Fredholm determinant denominator of (6) in one of the two equivalent forms

$$\mathfrak{D}(\mu) = \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n!} \times \int_{m/2}^{\infty} dx_1 \cdots \int_{m/2}^{\infty} dx_n \Delta(x_1)\cdots\Delta(x_n) P_n(x_1, \cdots x_n)$$

$$P_n(x_1, \cdots x_n) = \begin{vmatrix} \frac{1}{2x_1} & \frac{1}{x_1+x_2} & \cdots & \frac{1}{x_1+x_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{x_1+x_n} & & \cdots & \frac{1}{2x_n} \end{vmatrix}, \tag{8a}$$

$$\mathfrak{D}(\mu) = \exp\left[-\left(\sum_1^{\infty} \mu^n \frac{A_n}{n}\right)\right]$$

$$A_1 = \int_{m/2}^{\infty} dx_1 \frac{\Delta(x_1)}{2x_1};$$

$$A_n = \int_{m/2}^{\infty} dx_1 \cdots \int_{m/2}^{\infty} dx_n \times \frac{\Delta(x_1)\cdots\Delta(x_n)}{(x_1+x_2)(x_2+x_3)\cdots(x_n+x_1)}, \tag{8b}$$

where A_n are the traces of the kernel of (6). If we remember our above discussion, in order to take into account the fact that the solution certainly exists always, for instance, for Yukawa type of potentials, we have the feeling that $\mathfrak{D}(\mu)$ for $l = 0$, as well as for $l \neq 0$ must make explicit condition (1c). The main result of this paper is to show that for $l = 0$, $\mathfrak{D}(\mu)$ can in fact

[8] F. G. Tricomi, *Integral Equations* (Interscience Publishers, New York, 1957), Chap. II, p. 49.

[9] We do not consider here the ambiguity coming in this type of inverse problem from phase equivalent potentials [see K. Chadan, Nuovo Cimento **24**, 298 (1962) and A. Martin, Ref. 4].

[10] See the references given by R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1962), Vol. I, p. 161.

be reduced in a very simple closed form:

$$\mathfrak{D}(\mu) = \exp\left(-\int_{m/2}^{\infty} \frac{C(2x, \mu)\, dx}{(2x)^2}\right) = \exp\left(-\int_{m/2}^{\infty} \frac{B(x, \mu)}{2x}\right) dx, \tag{9}$$

where $C(\alpha, \mu)$ is the inverse Laplace transform of the potential reconstructed from the discontinuity $\mu\Delta(x)$ and $B(x, \mu) = C(2x, \mu)/2x$ is the part of the discontinuity coming from the first Born approximation. The most powerful and straightforward tool for the study of the solutions of Eqs. (6) and (7) or for the study of $\mathfrak{D}(\mu)$ is certainly the Marchenko[6] formalism as will be emphasized in the following. But we are interested in this paper in order to make explicit the connection between different formalisms so that we think it is also useful to show the connection between the existence of solutions of Eqs. (6) and (7) and the inverse problem of the determination of $C(\alpha, \mu)$ from $\mu\Delta(x)$ as given by the Martin relation.[4]

III. CONNECTION BETWEEN MARTIN'S INVERSION PROCEDURE AND $\mathfrak{D}(\mu)$

Martin[4] has given for Yukawa type of potentials the relation between $C(2x, \mu)$ [or $B(x, \mu)$] and $\mu\Delta(x)$. We recall[4] that for (1) the Jost solutions can be written as a Laplace transform

$$f(k, r) = e^{-ikr}\left[1 + \int_m^{\infty} e^{-\alpha r}\rho_k(\alpha)\, d\alpha\right] \underset{r\to\infty}{\simeq} e^{-ikr}$$

and $\rho_k(\alpha)$ satisfies a Volterra integral equation. Always for family (1) if we define $2(y - x)\rho_{ix}(2y) = \tau(x, y)$, $(y < x)$, then Martin has shown that

$$\lim_{y\to x} \tau(x, y) = \mu\Delta(x)$$

and the relations giving the possibility of reconstructing the potential from the knowledge of the discontinuity are

$$B(\mu, x) = \mu\Delta(x) - \frac{1}{x}\int_{m/2}^{x-m/2} B(\mu, x - y)\tau(x, y)\, dy,$$

$$\tau(x, y) = B(\mu, y) + \frac{1}{y}\int_{m/2}^{y-m/2} B(\mu, y - z)\tau(x, z)\frac{y - z}{z - x}\, dz, \qquad z < y < x. \tag{10}$$

From (10) it is easy to see that we can put

$$B(\mu, x) = \sum_{n-1}^{\infty} \mu^n B_n(x)\theta(x - \tfrac{1}{2}nm).$$

In order to obtain (9), it is equivalent to show that

$$\frac{1}{2}\int_{mn}^{\infty} \frac{C_n(\alpha)}{\alpha^2}\, d\alpha = \int_{nm/2}^{\infty} \frac{B_n(x)}{2x}\, dx = \frac{A_n}{n}, \tag{11}$$

where C_n is the nth term for the potential $B_n(x) = C_n(2x)/2x$. $\tau(x, y)$ can also be written as

$$\tau(x, y) = \sum_{n=1}^{\infty} \mu^n\tau_n(x, y)\vartheta(y - \tfrac{1}{2}nm).$$

Then, from (10), if we substitute these expansions we get two relations $B_n = H_1(B_1, \cdots B_{n-1}, \tau_1 \cdots \tau_{n-1})$ and $\tau_n = H_2(B_1, \cdots B_n, \tau_1, \cdots \tau_{n-1})$ such that if we know $B_1, \cdots B_{n-1}, \tau_1, \cdots \tau_{n-1}$, we can obtain B_n and τ_n as function of Δ. In this manner we get

$$B_1 = \tau_1 = \Delta,$$

$$B_2(x) = \frac{1}{x}\int_{m/2}^{x-m/2} \Delta(x - y)\Delta(y)\, dy,$$

$$B_3(x) = \int_m^{x-m/2} dy \int_{m/2}^{y-m/2} dz\, \frac{\Delta(x - y)\Delta(y - z)\Delta(z)}{xy} \times \frac{y + z - 2x}{z - x}. \tag{12}$$

From B_1, B_2, B_3 it is easy to verify (11) for 1, 2, 3. (See for instance Appendix A for B_3.) Unfortunately it is difficult from (10) to find for any n the explicit form of $B_n(x)$. However in the following section from Marchenko formalism we demonstrate the relation (9) such that in fact (11) is true for any n. [See the expression of $\mathfrak{D}(\mu)$ given by (8b).] Then the connection between Martin's relation and the Fredholm integral equation (6) given by dispersion relations becomes clear. When we reconstruct the potential $C(\alpha, \mu)$ from the one-to-one Martin's correspondence between $C(\alpha, \mu)$ and $\mu\Delta(x)$ then

$$C(\alpha, \mu) = \sum \mu^n C_n(\alpha)\theta(\alpha - nm)$$

and the nth iterative $C_n(\alpha)$ is such that

$$\int_{mn}^{\infty} \frac{C_n(\alpha)}{2\alpha^2}\, d\alpha$$

equals the nth trace of the kernel $\Delta(y)/(x + y)$. The circle of convergence in the μ plane of

$$\sum \mu^n \int_{nm}^{\infty} \frac{C_n(\alpha)}{2\alpha^2}\, d\alpha$$

is determined by the smallest modulus root of $\mathfrak{D}(\mu) = 0$. It will be shown in the following paper that these roots correspond to the reconstructed potentials becoming repulsive and singular as r^{-2} near the origin or

$$C(\alpha, \mu) \underset{\alpha\to\infty}{\simeq} \text{const}\ \alpha.$$

IV. EQUIVALENCE BETWEEN MARCHENKO FORMALISM AND DISPERSION RELATION IN THE f/f FORMULATION

In potential scattering we can consider two different formalisms in order to obtain the Jost function. The first one is derived from Green's functions formalisms where we get integral equations with kernels proportional to the potentials. In this manner we get for the Jost solution a Volterra integral equation directly in coordinate space or by using Laplace transform. In these cases for S wave and regular potentials the Fredholm determinant is 1.

If we remember our discussion in the Introduction, this corresponds to the second point of view: the case where the parameter is in fact the coupling constant λ of the potential $\lambda V(r)$ and where the integral equations are linear in λ. The second approach comes from dispersion relations and we get the resulting integral equation of the Fredholm type (4) or (6) with kernel proportional to the discontinuity. If we try to identify term by term the two formalisms, in principle we have only to replace the potential by its series coming from the discontinuity or the discontinuity by its series coming from the potential. [In Eq. (10) in one case we put the discontinuity as $\mu\Delta(x)$ and B or C as $B(x, \mu)$ or $C(\alpha, \mu)$; in the other case we put the discontinuity as $\Delta(x, \lambda)$ and B or C as $\lambda B(x)$ or $\lambda C(\alpha)$.] But in practice we have the same difficulty as in the preceding section—we can make the identification easily only for the first terms for both formulations. This type of technical difficulty is well known when we try to identify term by term two formulations, one given by dispersion relation and the other by the usual perturbation theory. But there exists another powerful formalism connecting at the same time-scattering data (discontinuity), potentials, and Jost function. This formalism: Marchenko[6] formalism uses a function of two variables $K(r, y)$, where $K(r, r)$ is linked to the primitive of the potential and $K(0, y)$ is the inverse Laplace of our Jost function $F(x) - 1$. The uniqueness of the inversion procedure is satisfied[6] if the potential is assumed to satisfy the usual conditions of moments finite. Furthermore $K(r, y)$ can be obtained from an integral equation where the kernel is the scattering data (proportional to the discontinuity). Then, we can, in Marchenko formalism, use the same linear parameter μ as for the resulting integral equation of f/f. Then it will not be very surprising if we can identify the two formalisms.

A. Marchenko Equations

We recall the Marchenko equations[6]

$$K(r, y) = \mathcal{F}(r + y) + \int_r^\infty K(r, t)\mathcal{F}(t + y)\, dt, \tag{13a}$$

$$F(x, r) = f(-ix, r) = e^{-xr} + \int_r^\infty K(r, y)e^{-xy}\, dy, \qquad F(x, 0) = F(x), \tag{13b}$$

$$V(r) = -2\frac{d}{dr}(K(r, r)), \tag{13c}$$

$$-\mathcal{F}(t) = \sum_{j=1}^{p} M_j^2 e^{-|x_j|t} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}[1 - S(k)]e^{ikt}dk, \tag{13d}$$

where M_j^2 are the normalization constants corresponding to the negative eigenvalues $-|x_j|^2$. We want to transform the scattering data (13d) in order to introduce the discontinuity.

For Yukawa-type family (1) we can rotate the integration path in the upper-half k plane ($\operatorname{Im} k > 0$) for the integral in (13d). Because for (1) the contribution along the half great circle ($|k| \to \infty$, $\operatorname{Im} k > 0$) is 0 then only the cut remains along the imaginary axis and the poles of the S matrix are also along the imaginary axis. The residues of these poles cancel the first part of the right-hand side of (13d) and we get

$$\mathcal{F}(t) = \int_{m/2}^\infty e^{-tu}\mu\Delta(u)\, du. \tag{13d$'$}$$

Substituting (13d′) and (13a) in (13b), we find for the Jost solution $F(x, r)$:

$$F(x, r) = e^{-xr} + \mu\int_{m/2}^\infty \frac{\Delta(y)e^{-r(x+y)}}{x + y} F(y, r)\, dy. \tag{14}$$

We see that for $r = 0$ the Jost function $F(x, 0) = F(x)$ satisfies exactly the same integral equation (6) as given by dispersion relation.

Because of the equivalence between the two formalisms, we adopt the same point of view as in the preceding sections. We consider formally Eqs. (13) with the only restriction that the potential is local [in fact we shall obtain a larger family than (1)] and assume only the regularity condition (7) for the discontinuity such that (13a) and (14) are of the Fredholm type. We still seek to interpret the solutions of (6) with the help of (13) and (14). Note that we do not assume the conditions of moments finite for the potentials such that now the determinant of (13a) and (14) can vanish.

We remark that the Fredholm determinant of (14) is

$$\mathfrak{D}(\mu, r) = 1 + \sum_{n=1}^\infty \frac{(-\mu)^n}{n!}\int_{m/2}^\infty du_1 \cdots \int_{m/2}^\infty du_n \times \left(\prod_{n=1}^n \Delta(u_i)e^{-2ru_i}\right)P_n(u_1, \cdots u_i, \cdots u_n), \tag{15a}$$

$$\begin{aligned}\mathfrak{D}(\mu, r) &= \exp\left(-\sum_{n=1}^\infty \mu^n \frac{A_n}{n}(r)\right),\\ A_1(r) &= \int_{m/2}^\infty e^{-2ru_1}\frac{\Delta(u_1)\, du_1}{2u_1},\\ A_n(r) &= \int_{m/2}^\infty du_1 \cdots \int_{m/2}^\infty du_n \times \frac{\prod_{n=1}^n \Delta(u_i)e^{-2ru_i}}{(u_1 + u_2)(u_2 + u_3)\cdots(u_n + u_1)}\end{aligned} \tag{15b}$$

such that for $r = 0$ we have $\mathfrak{D}(\mu, 0) = \mathfrak{D}(\mu)$.

In this paper and in the following we want to find general features for the solutions $K(r, r)$, $F(x, r)$, $F(x, 0)$, $F(0, r)$ of Eqs. (13), (14), and (6) when the condition (7) is assumed for the discontinuity.

B. An Example: The Discontinuity Replaced by a Simple Pole

As illustration we consider first the simple case $\mu\Delta = \mu\delta(x - b)$ "As has been known for some time"[11] the Jost–Bargman potential[12] leads to this type of discontinuity. From Eqs. (13) and (14) we get in this case

$$K(r, r) = \frac{\mu e^{-2br}}{1 - (\mu/2b)e^{-2br}},$$

$$F(x, r) = e^{-xr}\left[1 + \left(\frac{\mu e^{-2br}}{1 - (\mu/2b)e^{-2br}}\right)\frac{1}{b + x}\right].$$

We remark:

(a) The Fredholm determinants of (14) and (13a) are the same.

(b) $K(r, r)$ is written as a ratio where the numerator is the derivative of the denominator, this denominator being $\mathfrak{D}(\mu, r) = 1 - (\mu/2b)e^{-2br}$.

(c)

$$V(\mu, r) = 8b^2 \sum n\left(\frac{\mu}{2b}\right)^n e^{-2nbr}$$

and

$$B_n(x) = \frac{C_n(2x)}{2x} = \frac{4b^2n}{x}\,\delta(2x - 2bn).$$

We can then verify

$$A_n(r)/n = e^{-2brn}(2b)^{-n} = \int_{m/2}^{\infty} e^{-2xr}\,\frac{B_n(x)}{2x}\,dx$$

for any n and $r \geq 0$. [For $r = 0$ this is relation (11) of the previous section.] We have also

$$V(\mu, r) = 2\sum (\mu^n/n)\, d^2A_n(r)/dr^2.$$

(d) $\mu = 2b$ is the root of $\mathfrak{D}(\mu) = 0$. For $|\mu| < 2b$, then $\mathfrak{D}(\mu, r)$ cannot vanish; $V(\mu, r)$ is a regular potential without poles for $r \geq 0$; $F(x, r)$ has no poles for $r \geq 0$; the series of the traces $\sum \mu^n A_n(r)/n$ as well as the series

$$\frac{1}{2}\sum_n \mu^n \int_m^{\infty} \frac{C_n(\alpha)}{\alpha^2}\, e^{-\alpha r}$$

converge for $r \geq 0$.

(e) For $\mu = 2b$, $V(\mu, r)$ becomes singular and repulsive like r^{-2} at $r = 0$. For $\mu > 2b$, V has a pole of the second order in $r > 0$ and the corresponding "bound state" is a ghost with a pole in $r > 0$ for the ghost-wave solution.

For $\mu \leq -2b$ we never encounter other roots of $\mathfrak{D}(\mu)$ and $\mathfrak{D}(\mu, r)$, V has no poles for r on the real axis ($\mathrm{Re}\, r \geq 0$), the bound state is a true bound state.

We remark also that $F(0, r)$ can be written $\mathfrak{D}(-\mu, r)/\mathfrak{D}(\mu, r)$ showing the connection between roots of $\mathfrak{D}(\pm\mu, r)$, $(r \geq 0)$ and poles or roots of $F(0, r)$ $(r \geq 0)$—therefore, the connection between ghosts and bound states.

C. General Case: Fundamental Relation between $V(\mu, r)$ and $\mathfrak{D}(\mu)$

Now we come back to the general case: First with the bound given by De Alfaro and Regge[5]

$$|P_n(u_i)| < \prod_{n=1}^{n} \frac{1}{2u_i}$$

we get from (15a)

$$|\mathfrak{D}(\mu, r)| < e^{A(\mu, \mathrm{Re}\, r)},$$

$$|\mathfrak{D}(\mu, r) - 1| < e^{A(\mu, \mathrm{Re}\, r)} - 1 \leq e^{A(\mu, 0)} - 1, \quad (16)$$

$$A(\mu, \mathrm{Re}\, r) = \int_{m/2}^{\infty} \frac{|\mu\Delta(u)|}{2u}\, e^{-2\,\mathrm{Re}\, ru}\, du,$$

where $\mathrm{Re}\, r \geq 0$.

(a) It is shown in Appendix B that in the general case the Fredholm determinant $\mathfrak{D}(\mu, r)$ of (13a) and (14) are the same.

(b) It is shown in Appendix B that in the general case we have from (13a):

$$K(r, r) = \left(\frac{d}{dr}\,\mathfrak{D}(\mu, r)\right)\Big/\mathfrak{D}(\mu, r); \quad \mathfrak{D}(\mu, 0) = \mathfrak{D}(\mu). \quad (17)$$

From our fundamental relation (17) we get

$$\exp\left(-\frac{1}{2}\int_r^{\infty} dx \int_x^{\infty} V(\mu, t)\, dt\right) = \mathfrak{D}(\mu, r), \quad (18a)$$

$$\exp\left(-\frac{1}{2}\int_0^{\infty} dx \int_x^{\infty} V(\mu, t)\, dt\right) = \mathfrak{D}(\mu), \quad (18b)$$

where we have used the fact that, following the bound (16),

$$|\mathfrak{D}(\mu, r)| \underset{r\to\infty}{\longrightarrow} 1.$$

Because of (17), $K(r, r)$ for $r > 0$ can have poles at most of the first order [$\mathfrak{D}(\mu, r)$ is analytic in $\mathrm{Re}\, r > 0$], in these cases the integral in (18b) is taken as a Cauchy principal value. If the pole corresponds to an end-point singularity $r \neq 0$ in (18a) or $r = 0$ in (18b), then the corresponding $\mathfrak{D}(\mu, r)$ or $\mathfrak{D}(\mu)$ are equal to zero. The different cases $\mathfrak{D}(\mu, r) = 0$ or $\neq 0$ $(r \geq 0)$ are discussed here and in the following paper. If we write $V(\mu, r)$ as a Laplace transform, then

$$\frac{1}{2}\int_m^{\infty} e^{-\alpha r}\,\frac{C(\alpha, \mu)}{\alpha^2}\, d\alpha = -\log \mathfrak{D}(\mu, r) = \sum_{n=1}^{\infty} \mu^n \frac{A_n(r)}{n}\; r \geq 0. \quad (19)$$

For $r = 0$ this is the relation (9).

[11] L. Heller and M. Rich, Phys. Rev. **144**, 1324 (1966).

[12] R. Jost, Helv. Phys. Acta **20**, 256 (1947); V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949); Phys. Rev. **75**, 301 (1949).

(c) The relations (18a) and (15b) give us the possibility of writing $V(\mu, r)$ as a series in μ where the coefficients depend on $\Delta(x)$. Because of our fundamental relation (17), then $V = 2[(\mathfrak{D}')^2 - \mathfrak{D}\mathfrak{D}'']/\mathfrak{D}^2$. For $r = r_0 > 0$ fixed, these expansions will have circles of convergence given by the smallest modulus root of $\mathfrak{D}(\mu, r_0)$,

$$V(\mu, r) = 2\sum_{n=1}^{\infty} \frac{\mu^n}{n} \frac{d^2 An(r)}{dr^2}, \quad r > 0, \tag{20}$$

where $A_n(r)$ is given by (15b). [We give the explicit form of (20) in the introduction.]

If we consider the nth term of the inverse Laplace transform $C(\alpha, \mu) = \sum \mu_n C_n(\alpha)$, we get from Eq. (19)

$$\frac{1}{2}\int_m^{\infty} e^{-\alpha r} \frac{C_n(\alpha)}{\alpha^2}\, d\alpha = \frac{A_n(r)}{n}, \quad r \geq 0. \tag{21}$$

For $r = 0$ this is the relation (11) of the above section.

When we take the inverse Laplace transform of (21) we can get different equivalent expressions depending upon whether we use $A_n(r)$ or $A_n'(r)$ or $A_n''(r)$. For instance with $A_n'(r)$ we get

$$\begin{aligned}\frac{C_n(2x)}{2x} &= B_n(x)\\ &= \int_{(n-1)m/2}^{x-m/2} du_1 \int_{(n-2)m/2}^{u_1-m/2} du_2 \cdots \int_{m/2}^{u_{n-2}-m/2} du_{n-1}\\ &\times \frac{\Delta(x-u_1)\Delta(u_1-u_2)\cdots\Delta(u_{n-2}-u_{n-1})\,\Delta(u_{n-1})}{(x-u_2)(u_1-u_3)(u_2-u_4)\cdots(u_{n-3}-u_{n-2})u_{n-2}}.\end{aligned} \tag{22}$$

Note that because of the ranges of integration, the denominator in (22) cannot vanish.

First we observe from (22) that $C_n(2x) = 0$ if $2x < nm$; second, (22) is the solution for any n of the equations (10) where we were unable to find directly the general term B_n [In Appendix A it is shown for instance that (22) for $n = 3$ gives the same result as B_3 given by (12).] Third, from the explicit determination of the nth term $C_n(\alpha)$ given by (22) it is easy to verify the property previously given by Martin[4]: if $\Delta(x)$ is known up to $x = x_{\max}$, then $C(\alpha, \mu)$ is known up to $\alpha = 2x_{\max}$ and the reverse is also true.

(d) First for $|\mu|$ sufficiently small, the De Alfaro and Regge condition can be satisfied if $A(\mu, 0) < \log 2$. In this case we see from the bounds (16) that both $\mathfrak{D}(\mu, r)$ $(r > 0)$ and $\mathfrak{D}(\mu, 0)$ cannot vanish. $V(r, \mu)$ has no poles for $r \geq 0$ and the corresponding solutions $F(x, r)$ $(r \geq 0)$ of (14) and (6) are unique and exist always.

Second, these results are probably true for larger $|\mu|$ values. We assume that $\Delta(x)$ has only a finite number of changes of signs. We call μ_{+j} and μ_{-j} the positive and negative roots of $\mathfrak{D}(\mu)$ and ($\mu_j < \mu_{j+1}$ $|\mu_{-j}| < |\mu_{-(j+1)}|$). We consider

$$|\mu| < \inf(\mu_1, |\mu_{-1}|).$$

In this case, the series of the traces [of (6) or (14) for $r = 0$] $\sum \mu^n(A_n/n)$ $(r = 0)$ converge following (19) because the circle of convergence is determined by the first smallest modulus root of $\mathfrak{D}(\mu) = 0$. In this case because of (19), as long as $C(2x, \mu)$ exists for $|\mu| < |\mu|_C$,

$$\left|\int_{m/2}^{\infty} \frac{C(2x, \mu)}{x^2}\, dx\right| < \infty$$

and the potential is "regular" at the origin. [This follows also from the fact that the solution

$$F(x, r) \underset{r\to 0}{\longrightarrow} \text{const}$$

when $\mathfrak{D}(\mu) \neq 0$.] But because $C(\alpha, \mu)/\alpha^2$ is integrable then $e^{-\alpha r}C(\alpha, \mu)/\alpha^2$ is also integrable for $r > 0$ and

$$\left|\int_m^{\infty} e^{-\alpha r} \frac{C(\alpha, \mu)}{\alpha^2}\, d\alpha\right| < \infty.$$

It follows from (19) also that $\mathfrak{D}(\mu, r)$ cannot vanish in the same range $|\mu| < |\mu|_C$ and the series of the traces (19) for $r > 0$ of the integral equation (19) converge also. Furthermore in the same $|\mu|$ range, $V(r, \mu)$ has no poles for $r > 0$ and the expansion (20) of the potential converges. In conclusion this range is characterized by

$$\left|\int_m^{\infty} e^{-\alpha r} \frac{C(\alpha, \mu)}{\alpha^2}\right| < \infty, \quad r \geq 0.$$

(e) The cases $|\mu|$ larger than $\inf(\mu_1, |\mu_{-1}|)$ will be studied in the following paper[13] for the solution of (14) and (6) as well as the interpretation in terms of $V(\mu, r)$ and the connection between bound states and ghost. We will find in general that the roots $\mu_{\pm j}$ correspond to $V(\mu, r)$ becoming repulsive and singular like r^{-2} at the origin. We shall also find a larger domain in μ (not $|\mu|$) where the solutions $F(x, r)$ $(r \geq 0)$ exist in the Fredholm form with acceptable physical meaning.

D. Case Where $\mu\Delta(x) = \Delta(x, \lambda)$

In this section we have assumed that the discontinuity is $\mu\Delta(x)$ with μ as a linear parameter. But, as we have said in the introduction, we can adopt another point of view where the discontinuity is $\Delta(x, \lambda)$ given by the Born series $\sum \lambda^\rho \Delta_\rho(x)$, and λ is the coupling parameter [for instance, for family (1)] of the potential. In this case we write

$$\mathcal{F}(t) = -\int_{m/2}^{\infty} e^{-tu}\Delta(u, \lambda)\, du$$

[13] H. Cornille, J. Math. Phys. 8 (to be published.)

and we still obtain the integral equation (14) or our fundamental relation (17), where $\mu\Delta$ is replaced by $\Delta(x, \lambda)$. In this case the Fredholm denominator of (14) and (6) is

$$\exp\left(-\frac{1}{2}\int_m^\infty \lambda e^{-\alpha r}\frac{C(\alpha)}{\alpha^2}\,d\alpha\right), \quad r \geq 0.$$

This shows for instance [note that

$$\left|\int_m^\infty \lambda\,\frac{C(\alpha)}{\alpha^2}\,d\alpha\right| < \infty$$

is slightly weaker than (1c)] that for Yukawa-type potentials (1) the solution of the resulting integral equation written in the Fredholm type exists for any values of the coupling constant λ. We want to emphasize that with this point of view, the potential or the first Born discontinuity is the input and contrary to the other case we have not of course to study the properties of the reconstructed potential.

V. CONCLUSION

In this paper we have shown that the resulting integral equation of the f/f equation can also be obtained from Marchenko formalism. This fact gives the possibility to seek for the existence of the solutions and to interpret their meaning for "regular discontinuities."

First if the whole discontinuity is linear with respect to the parameter μ, then for $|\mu| < \inf(\mu_1, |\mu_{-1}|)$ the solutions exist always and $V(\mu, r)$ is regular at $r = 0$. The case $|\mu| > \inf(\mu_1, |\mu_{-1}|)$ will be discussed in a following paper.[13]

Secondly, if we consider the discontinuity as given by the Born series $\sum \lambda^p\Delta_p(x)$ with λ coupling constant of a regular potential (like Yukawa type) or coupling constant of the corresponding first Born term, then the solution of the resulting integral equation exists always.

ACKNOWLEDGMENTS

The author thanks Professor L. Schwartz and the theoreticians of the Theoretical Division for their interest in this work.

APPENDIX A

We want to verify

$$\int_{3m/2}^\infty \frac{B_3(x)}{2x} = \tfrac{1}{2}A_3,$$

where B_3 is given by (12). This can be made directly but we want also to verify the equality of B_3 given by (12) and (22). In (12) we put $V = x - z$, $W = x - y$ and after we put $V = y$, $W = z$ such that we get

$$B_3(x) = \int_m^{x-m/2} \Delta(x-y) \times \int_{m/2}^{y-m/2} \frac{\Delta(y-z)\Delta(z)(y+z)}{xy(x-z)}\,dz\,dy.$$

Now we add to this result B_3 given by (12) and we get

$$B_3(x) = \frac{C_3(2x)}{2x} = \int_m^{x-m/2} \Delta(x-y)\int_{m/2}^{y-m/2} \frac{\Delta(y-z)\Delta(z)\,dz\,dy}{y(x-z)}.$$

This is the result obtained from Marchenko formalism (22). Now we put $x = x_1 + x_2 + x_3$, $y = x_2 + x_3$, $z = x_3$ and we get

$$\int_{3m/2}^\infty \frac{B_3(x)}{2x}\,dx = \int_{m/2}^\infty\int_{m/2}^\infty\int_{m/2}^\infty \frac{\Delta(x_1)\Delta(x_2)\Delta(x_3)\,dx_1\,dx_2\,dx_3}{2(x_1+x_2)(x_2+x_3)(x_1+x_2+x_3)}$$

and

$$0 = \int_{3m/2}^\infty \frac{B_3(x)}{2x}\,dx - \frac{A_3}{3} = \frac{1}{6}\int_{m/2}^\infty dx_3\int_{m/2}^\infty dx_2\int_{m/2}^\infty dx_1\,\frac{\Delta(x_1)\Delta(x_2)\Delta(x_3)[3x_1 + 3x_3 - 2x_1 - 2x_2 - 2x_3]}{(x_1+x_2+x_3)(x_1+x_2)(x_2+x_3)(x_3+x_1)}.$$

APPENDIX B: DETERMINATION OF $K(r, r)$

We study the Fredholm solution $K(r, y)$ for $y = r$ of (13a) with the scattering data given by (13d'). We put $\mathscr{F}(t) = \mu G(t)$.

We want to show that the solution $K(r, y = r)$ of

$$K(r, y) = \mu G(r + y) + \mu\int_r^\infty K(r, t)G(y + t)\,dt \tag{B1}$$

with

$$G(y + t) = \int_{m/2}^\infty e^{-(y+t)u}\Delta(u)\,du$$

can be written

$$K(r, r) = \left(\frac{d}{dr}\mathfrak{D}(\mu, r)\right)\Big/\mathfrak{D}(\mu, r),$$

where $\mathfrak{D}(\mu, r)$ is given by (15a).

1. The Fredholm determinant of (B1) and (14) are the same. The Fredholm determinant $\overline{\mathfrak{D}}(\mu, r)$ of (B1) is

$$\overline{\mathfrak{D}}(\mu, r) = 1 + \sum_{n=1}^{\infty}\frac{(-\mu)^n}{n!}\int_r^\infty dr_1 \cdots \int_r^\infty dt_n \begin{vmatrix} G(2t_1) & G(t_1+t_2) & \cdots & G(t_1+t_n) \\ G(t_2+t_1) & & & \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ G(t_n+t_1) & & \cdots & G(2t_n) \end{vmatrix}$$

$$\overline{\mathfrak{D}}(\mu, r) = 1 + \sum_{n=1}^{\infty}\frac{(-\mu)^n}{n!}\int_r^\infty dt_1 \cdots \int_r^\infty dt_n \int_{m/2}^\infty du_1 \cdots \int_{m/2}^\infty du_n Q_n(t_i, u_i)\left(\prod_{n=1}^{n}\Delta(u_i)\right),$$

where

$$Q_n = \begin{vmatrix} e^{-2t_1u_1} & e^{-(t_1+t_2)u_1} & \cdots & e^{-(t_1+t_n)u_1} \\ e^{-(t_1+t_2)u_2} & & & \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ e^{-(t_1+t_n)u_n} & & \cdots & e^{-2t_nu_n} \end{vmatrix} = \begin{vmatrix} e^{-2t_1u_1} & e^{-t_2(u_1+u_2)} & \cdots & e^{-t_n(u_1+u_n)} \\ e^{-t_1(u_1+u_2)} & e^{-2t_2u_2} & & \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e^{-t_1(u_1+u_n)} & & \cdots & e^{-2t_nu_n} \end{vmatrix}.$$

Then

$$\overline{\mathfrak{D}}(\mu, r) = 1 + \sum_{n=1}^{\infty}\frac{(-\mu)^n}{n!}\int_{m/2}^\infty du_1 \cdots \int_{m/2}^\infty du_n\left(\prod_{n=1}^{n}\Delta(u_i)\right)\begin{vmatrix} \frac{e^{-2ru_1}}{2u_1} & \frac{e^{-r(u_1+u_2)}}{u_1+u_2} & \cdots & \frac{e^{-r(u_1+u_n)}}{u_1+u_n} \\ \frac{e^{-r(u_1+u_2)}}{u_1+u_2} & & \cdots & \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \frac{e^{-r(u_1+u_n)}}{u_1+u_n} & & \cdots & \frac{e^{-2ru_n}}{2u_n} \end{vmatrix} \tag{B2}$$

or

$$\overline{\mathfrak{D}}(\mu, r) = 1 + \sum_{n=1}^{\infty}\frac{(-\mu)^n}{n!}\int_{m/2}^\infty du_1 \cdots \int_{m/2}^\infty du_n P_n(u_1, \cdots u_n)\left(\prod_{n=1}^{n}\Delta(u_i)e^{-2ru_i}\right), \tag{B3}$$

where P_n is given by (8a). The result (B3) shows that $\overline{\mathfrak{D}}(\mu, r) = \mathfrak{D}(\mu, r)$ and that for $r = 0$, $\overline{\mathfrak{D}}(\mu, 0) = \mathfrak{D}(\mu, 0) = \mathfrak{D}(\mu)$ as given by (8a).

From (B2) we can write $(d/dr)\mathfrak{D}(\mu, r)$ as

$$\mathfrak{D}'(\mu, r) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\mu^n\int_{m/2}^\infty du_1 \cdots \int_{m/2}^\infty du_n\left(\prod_{n=1}^{n}\Delta(u_i)e^{-2ru_i}\right)\left(\sum_{k=1}^{n}E_k(u_1, \cdots u_n)\right).$$

E_k is the determinant P_n where the kth row is replaced by $1 \cdots 1 \cdots 1$:

$$E_k = \begin{vmatrix} \frac{1}{2u_1} & \frac{1}{u_1+u_2} & \cdots & \frac{1}{u_1+u_n} \\ & \cdot & \cdots & \cdot \\ \frac{1}{u_1+u_{k-1}} & \cdot & \cdots & \cdot \\ & \cdot & \cdots & \cdot \\ 1 & 1 & \cdots & 1 \\ \frac{1}{u_1+u_{k+1}} & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \frac{1}{u_1+u_n} & & \cdots & \frac{1}{2u_n} \end{vmatrix}.$$

We want to show that for any k

$$H_k = \int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=1}^{n} \Delta(u_i) e^{-2ru_i}\right) E_k$$

are equal—for instance, $H_k = H_1$. We remark that u_k appears only in the kth column.

For this in E_k we exchange the first and the kth row and after that, the first and kth column. Then u_1 appears only in the kth row and in the kth column,

$$E_k = \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ \frac{1}{u_2 + u_k} & \frac{1}{2u_2} & \cdots & \frac{1}{u_2 + u_1} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \frac{1}{u_{k-1} + u_k} & & \cdots & \frac{1}{u_{k-1} + u_1} & \cdots & \cdot \\ \frac{1}{u_1 + u_k} & \frac{1}{u_1 + u_2} & \cdots & \frac{1}{2u_1} & \cdots & \frac{1}{u_1 + u_n} \\ \frac{1}{u_n + u_k} & & \cdots & \frac{1}{u_n + u_1} & \cdots & \frac{1}{2u_n} \end{vmatrix}.$$

u_k appears only in the first column. Because the factor $\prod_i \Delta(u_i)e^{-2ru_i}$ is symmetric with respect to all the variables u_i and does not change if we put $u_i = u_k$ and $u_k = u_1$, then by this change $E_k = E_1$ and $H_k = H_1$.

Finally,

$$\frac{d}{dr}(\mathfrak{D}(\mu, r)) = \sum (-1)^{n+1} \frac{\mu^n}{(n-1)!} \int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=1}^{n} \Delta(u_i) e^{-2ru_i}\right) E_1(u_1, \cdots u_n),$$

$$E_1 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{u_1 + u_2} & \frac{1}{2u_2} & \cdots & \frac{1}{u_2 + u_n} \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \frac{1}{u_1 + u_n} & & \cdots & \frac{1}{2u_n} \end{vmatrix}. \tag{B4}$$

2. The Fredholm solution of (B1) can be written

$$K(r, y) = \mu G(r + y) + \frac{\mu^2 \int_r^{\infty} N_r(y, t) G(r + t)\, dt}{\mathfrak{D}(\mu, r)},$$

where $N_r(y, t)$ is the Fredholm numerator determinant of (B1):

$$N_r(y, t) = G(y + t) + \sum_1^{\infty} \frac{(-\mu)^n}{n!} \int_r^{\infty} dt_1 \cdots \int_r^{\infty} dt_n \begin{vmatrix} G(y + t) & G(y + t_1) & \cdots & G(y + t_n) \\ G(t_1 + t) & G(2t_1) & \cdots & G(t_1 + t_n) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ G(t_n + t) & G(t_n + t_1) & \cdots & G(2t_n) \end{vmatrix}.$$

$K(r, r)$ can be written

$$K(r, r) = N(\mu, r)/\mathfrak{D}(\mu, r),$$

where

$$\begin{aligned} N(\mu, r) &= \bar{N}(\mu, r) + \bar{\bar{N}}(\mu, r), \\ \bar{N}(\mu, r) &= \mu \mathfrak{D}(\mu, r) G(2r), \\ \bar{\bar{N}}(\mu, r) &= \mu^2 \int_r^{\infty} N_r(r, t) G(r + t)\, dt. \end{aligned} \tag{B5}$$

For $\bar{N}(\mu, r)$ we use $\mathfrak{D}(\mu, r)$ given by (B3) and we get

$$\bar{N}(\mu, r) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \mu^n \int_{m/2}^{\infty} e^{-2ru_1}\Delta(u_1)\, du_1 \int_{m/2}^{\infty} du_2 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=2}^{n} \Delta(u_i) e^{-2ru_i}\right) P_{n-1}(u_2, \cdots u_n)$$

or

$$\bar{N}(\mu, r) = \sum_{n=1}^{\infty} \frac{\mu^n(-1)^{n-1}}{(n-1)!} \int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{1}^{n} \Delta(u_i) e^{-2ru_i}\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & \dfrac{1}{2u_2} & \cdots & \dfrac{1}{u_2+u_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \dfrac{1}{u_2+u_n} & \cdots & \dfrac{1}{2u_n} \end{vmatrix}. \tag{B6}$$

We note that the determinant in (B6) is the same as P_n except that all the elements of the first row are 1 and the elements of the first column are $1, 0 \cdots 0 \cdots 0$. Now we study $\bar{\bar{N}}(\mu, r)$:

$$\bar{\bar{N}}(\mu, r) = \sum_{n=2}^{\infty} \frac{(-1)^{n-2}}{(n-2)!} \mu^n \int_r^{\infty} dt \int_r^{\infty} dt_1 \cdots \int_r^{\infty} dt_{n-2} \int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=1}^{n} \Delta(u_i)\right) Z_n(r, t, t_1, \cdots t_{n-2}, u_1, \cdots u_n),$$

where

$$Z_n = e^{-(r+t)u_1} \begin{vmatrix} e^{-(r+t)u_2} & e^{-(r+t_1)u_2} & \cdots & e^{-(r+t_{n-2})u_2} \\ e^{-(t_1+t)u_3} & e^{-2t_1u_3} & \cdots & e^{-(t_1+t_{n-2})u_3} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e^{-(t_{n-2}+t)u_n} & e^{-(t_{n-2}+t_1)u_n} & \cdots & e^{-2t_{n-2}u_n} \end{vmatrix}.$$

But we have also

$$Z_n = e^{-r(u_1+u_2)} \begin{vmatrix} e^{-t(u_1+u_2)} & e^{-t_1(u_3+u_2)} & \cdots & e^{-t_{n-2}(u_n+u_2)} \\ e^{-t(u_1+u_3)} & e^{-2t_1u_3} & \cdots & e^{-t_{n-2}(u_n+u_3)} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e^{-t(u_1+u_n)} & e^{-t_1(u_3+u_n)} & \cdots & e^{-2t_{n-2}u_n} \end{vmatrix}, \tag{B7}$$

and we get

$$\bar{\bar{N}}(\mu, r) = \sum_{2}^{\infty} \frac{(-1)^{n-2}\mu^n}{(n-2)!} \int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=1}^{n} \Delta(u_i) e^{-2ru_i}\right) L_n(u_1, u_2, \cdots u_n),$$

where L_n is the same determinant as P_n but with the first row and the second column lacking

$$L_n = \begin{vmatrix} \dfrac{1}{u_1+u_2} & \dfrac{1}{u_3+u_2} & \cdots & \dfrac{1}{u_n+u_2} \\ \dfrac{1}{u_1+u_3} & \dfrac{1}{2u_3} & \cdots & \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \dfrac{1}{u_1+u_n} & & \cdots & \dfrac{1}{2u_n} \end{vmatrix}.$$

Now we consider

$$A = \int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=1}^{n} \Delta(u_i) e^{-2ru_i}\right) M_n(u_1, \cdots u_n), \tag{B8}$$

with

$$M_n = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ \dfrac{1}{u_1+u_2} & \dfrac{1}{2u_2} & \dfrac{1}{u_1+u_3} & \cdots & \dfrac{1}{u_2+u_n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \dfrac{1}{u_1+u_n} & & & \cdots & \dfrac{1}{2u_n} \end{vmatrix} = \sum_{n=2}^{n}(-1)^{k+1}M_{n,k},$$

M_n is the same determinant as P_n except that all the elements of the first row are 0, 1, 1, $\cdots$, 1. We develop M_n following the elements of the first row, where we call $M_{n,k}$ the minor corresponding to the kth element of the first row. We remark that $L_n = M_{n,2}$. We want to show that

$$\int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=1}^{n} \Delta(u_i)e^{-2ru_i}\right)[M_{n,k}(u_1, \cdots, u_n) + (-1)^{k-1}M_{n,2}(u_1, \cdots, u_n)] = 0. \tag{B9}$$

For this, in $M_{n,k}$, we make the following substitutions: row $1 \to$ row 2, row $2 \to$ row 3, $\cdots$, row $k-2 \to$ row $k-1$, row $k-1 \to$ row 1. Then we get a new determinant $\bar{M}_{n,k} = (-1)^k M_{n,k}$ and

$$\bar{M}_{n,k} = \begin{vmatrix} \dfrac{1}{u_1+u_k} & \dfrac{1}{u_2+u_k} & \cdots & \dfrac{1}{u_k+u_{k-1}} & \dfrac{1}{u_k+u_{k+1}} & \cdots & \dfrac{1}{u_k+u_n} \\ \dfrac{1}{u_1+u_2} & \dfrac{1}{2u_2} & \cdots & \dfrac{1}{u_2+u_{k-1}} & \dfrac{1}{u_2+u_{k+1}} & \cdots & \dfrac{1}{u_2+u_n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \dfrac{1}{u_1+u_{k-1}} & \dfrac{1}{u_2+u_{k-1}} & \cdots & \dfrac{1}{2u_{k-1}} & \dfrac{1}{u_{k-1}+u_{k+1}} & \cdots & \dfrac{1}{u_{k-1}+u_n} \\ \dfrac{1}{u_1+u_{k+1}} & \dfrac{1}{u_2+u_{k+1}} & \cdots & \dfrac{1}{u_{k-1}+u_{k+1}} & \dfrac{1}{2u_{k+1}} & \cdots & \dfrac{1}{u_{k+1}+u_n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \dfrac{1}{u_1+u_n} & \dfrac{1}{u_2+u_n} & \cdots & \cdot & \cdot & \cdots & \dfrac{1}{2u_n} \end{vmatrix}.$$

Then, using the fact that the factor of $M_{n,k}$ in the integrand of (B9) is symmetric with respect to all the variables u_i, we put $u_k = u_2$, $u_2 = u_3$, $u_3 = u_4$, $\cdots$ $u_{k-1} = u_k$ and we get the relation (B9). Taking into account the relations (B8) and (B9) in (B7) we get

$$\bar{\bar{N}}(\mu, r) = \sum_{n=2}^{\infty} \mu^n \frac{(-1)^{n-1}}{(n-1)!} \int_{m/2}^{\infty} du_1 \cdots \int_{m/2}^{\infty} du_n \left(\prod_{n=1}^{n} \Delta(u_i)e^{-2ru_i}\right) \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ \dfrac{1}{u_1+u_2} & \dfrac{1}{2u_2} & & \cdots & \dfrac{1}{u_2+u_n} \\ \cdot & \cdot & & \cdots & \cdot \\ \cdot & \cdot & & \cdots & \cdot \\ \cdot & \cdot & & \cdots & \cdot \\ \dfrac{1}{u_1+u_n} & & & \cdots & \dfrac{1}{2u_n} \end{vmatrix}. \tag{B10}$$

In (B6) and (B10) the two determinants are the same except the first column. If we add these two determinants we get [see (B4)]

$$\bar{N}(\mu, r) + \bar{\bar{N}}(\mu, r) = (d/dr)\mathfrak{D}(\mu, r)$$

and finally

$$K(r, r) = \frac{(d/dr)\mathfrak{D}(\mu, r)}{\mathfrak{D}(\mu, r)}.$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Existence of Solutions of Crossing-Symmetric N/D Equations*

DAVID ATKINSON
Department of Physics, University of California, Berkeley, California

(Received 3 March 1967)

A fixed-point theorem is applied to the N/D equations of $\pi\pi$ scattering, for which crossing is satisfied exactly by the absorptive part of the amplitude, up to a finite, but arbitrary cut off. It is found that ghost-free solutions exist, if the subtraction constants are not too large, but that these solutions are not unique, since the Castillejo–Dalitz–Dyson ambiguity is not resolved by the requirement of crossing symmetry.

1. INTRODUCTION

THIS work is an initial attempt to answer the question as to whether the S-matrix equations[1] have solutions. In the following pages, the partial-wave N/D equations for the system $\pi\pi \rightarrow \pi\pi$,[2] with full crossing symmetry for the absorptive parts of the amplitudes up to a finite, but arbitrary cut off, are considered from the standpoint of a fixed-point theorem.[3] It is found that solutions exist, if the Castillejo–Dalitz–Dyson (CDD)[4] pole inhomogeneities are not too large, but that these CDD ambiguities are not resolved by the requirement of crossing symmetry.

It has been suggested in the past that the S-matrix requirements may be so stringent that no solutions exist, and that soluble simple models are impermissible approximations of the exact equations. The results of this paper go some small way towards a resolution of this uncertainty. The crossing-symmetric N/D equations do have solutions, but these possess the CDD ambiguity. Hence one might expect simple "bootstrap" models to make some limited sense; but one should not necessarily expect them to be free from all undetermined parameters.

The fact that at least one solution of the $\pi\pi$ equations exists should not be a surprise to readers of the numerical work of Chew, Mandelstam, and Noyes.[5] Their "S-dominant" solution is an example of the regular type of solution that is the concern of this paper. Nothing will be said for, or against the existence of a "P-dominant" singular type of solution.[6] However, that is not to say that the present work is necessarily divorced from the undoubted existence of the ρ-meson. It is possible that the "physical" solution is a regular solution containing a CDD pole in the P wave; and reasons for expecting this are collated in the Conclusion of this paper. This would mean that the ρ-meson parameters are incalculable in a one-channel $\pi\pi$ system, although one may be able to calculate them from suitable many-channel equations.

In an interesting paper,[7] Lovelace showed that if the crossing-symmetric partial-wave equations possess a solution, then it is locally unique if there are no CDD poles, and not unique if there are CDD poles.

* Research supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, under Grant No. AF-AFOSR-232-66, January 1967.

[1] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).
[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[3] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations* (The Macmillan Company, New York, 1964).
[4] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).
[5] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960).
[6] G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961).
[7] C. Lovelace, "Tests for the Uniqueness of the Physical Universe: An Application of Nonlinear Functional Analysis to S-Matrix Theory," CERN Preprint TH 689 (1966).

In (B6) and (B10) the two determinants are the same except the first column. If we add these two determinants we get [see (B4)]

$$\bar{N}(\mu, r) + \bar{\bar{N}}(\mu, r) = (d/dr)\mathfrak{D}(\mu, r)$$

and finally

$$K(r, r) = \frac{(d/dr)\mathfrak{D}(\mu, r)}{\mathfrak{D}(\mu, r)}.$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Existence of Solutions of Crossing-Symmetric N/D Equations*

DAVID ATKINSON
Department of Physics, University of California, Berkeley, California

(Received 3 March 1967)

A fixed-point theorem is applied to the N/D equations of $\pi\pi$ scattering, for which crossing is satisfied exactly by the absorptive part of the amplitude, up to a finite, but arbitrary cut off. It is found that ghost-free solutions exist, if the subtraction constants are not too large, but that these solutions are not unique, since the Castillejo–Dalitz–Dyson ambiguity is not resolved by the requirement of crossing symmetry.

1. INTRODUCTION

THIS work is an initial attempt to answer the question as to whether the S-matrix equations[1] have solutions. In the following pages, the partial-wave N/D equations for the system $\pi\pi \rightarrow \pi\pi$,[2] with full crossing symmetry for the absorptive parts of the amplitudes up to a finite, but arbitrary cut off, are considered from the standpoint of a fixed-point theorem.[3] It is found that solutions exist, if the Castillejo–Dalitz–Dyson (CDD)[4] pole inhomogeneities are not too large, but that these CDD ambiguities are not resolved by the requirement of crossing symmetry.

It has been suggested in the past that the S-matrix requirements may be so stringent that no solutions exist, and that soluble simple models are impermissible approximations of the exact equations. The results of this paper go some small way towards a resolution of this uncertainty. The crossing-symmetric N/D equations do have solutions, but these possess the CDD ambiguity. Hence one might expect simple "bootstrap" models to make some limited sense; but one should not necessarily expect them to be free from all undetermined parameters.

The fact that at least one solution of the $\pi\pi$ equations exists should not be a surprise to readers of the numerical work of Chew, Mandelstam, and Noyes.[5] Their "S-dominant" solution is an example of the regular type of solution that is the concern of this paper. Nothing will be said for, or against the existence of a "P-dominant" singular type of solution.[6] However, that is not to say that the present work is necessarily divorced from the undoubted existence of the ρ-meson. It is possible that the "physical" solution is a regular solution containing a CDD pole in the P wave; and reasons for expecting this are collated in the Conclusion of this paper. This would mean that the ρ-meson parameters are incalculable in a one-channel $\pi\pi$ system, although one may be able to calculate them from suitable many-channel equations.

In an interesting paper,[7] Lovelace showed that if the crossing-symmetric partial-wave equations possess a solution, then it is locally unique if there are no CDD poles, and not unique if there are CDD poles.

* Research supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, under Grant No. AF-AFOSR-232-66, January 1967.

[1] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[3] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations* (The Macmillan Company, New York, 1964).

[4] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[5] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960).

[6] G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961).

[7] C. Lovelace, "Tests for the Uniqueness of the Physical Universe: An Application of Nonlinear Functional Analysis to S-Matrix Theory," CERN Preprint TH 689 (1966).

The present work agrees with Lovelace's results; but it goes further, in that it proves the existence that was an explicit assumption of the earlier work. This is not a trivial point, for while there is no doubt that the universe exists (Wittgenstein notwithstanding), it is not clear that, at some level of approximation, it is a solution of the S-matrix equations.

In this paper, only the one-channel $\pi\pi$ system will be considered, although an inelasticity factor will be included. For convenience, it is assumed that this inelasticity is everywhere bounded, and that it satisfies certain smoothness conditions to be specified later. These conditions could certainly be relaxed. It is known that the effect of other channels may not be fully incorporated in the inelasticity factor, and that one-channel CDD poles may be required.[8,9,10] The many-channel system, as well as the complications of spin, will be considered in a later work, but it is expected that the same general results will apply.

In Sec. 2, the crossing-symmetric N/D equations are reformulated in such a way that infinite scattering lengths are excluded, and the CDD ambiguity is introduced so that the D function always remains finite in the physical region. This reformulation is a convenience for the ensuing proof. In Secs. 3 and 4, it is shown that the equations comprise a bounded, continuous, nonlinear mapping of a set of functions in a Banach space into a compact subset of the space. Under these conditions, the Schauder fixed-point principle, which is explained in Appendix A, asserts the existence of a solution. To aid the reader in a preliminary comprehension of the outlines of the proof, a few nonrigorous remarks concerning an iterative method of solution may be in order. The work of Sec. 3 may be regarded as a demonstration that, if the starting function for an iteration of the crossing-symmetric equations is not too large, then every iterate is bounded. In fact, each iterate may be represented by a point, its norm, in this case simply its maximum value. According to the proof of Sec. 3, the infinite number of points representing an infinite iteration would all be contained in a finite segment of the real line. By the Bolzano–Weierstrass theorem, there must be at least one point of accumulation on this line segment. Sec. 4 is devoted to the demonstration that at least one such point of accumulation is in fact the image of a solution of the system. To be able to assert that there is a "point of accumulation" in the Banach space of iterates, just as there is one on the line segment on which the norms were plotted, one must show that the iterates belong to a compact subset of the space. This is done in Sec. 4. However, this is not enough to demonstrate the existence of a solution, for there might be two points of accumulation in the space, and the iteration might jump alternately from the vicinity of one point to that of the other. Under these conditions one would not have found a solution. However, this possibility can be excluded by showing that the equations comprise the action of a *continuous* operator. This demonstration is also contained in Sec. 4. In Sec. 5, it is shown that, if the CDD parameters satisfy certain restrictions, then the partial-wave amplitudes have no ghosts.

[8] E. J. Squires, Nuovo Cimento **34**, 1751 (1964).

[9] M. Bander, P. W. Coulter, and G. L. Shaw, Phys. Rev. Letters **14**, 270 (1965).

[10] D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (N.Y.) **37**, 77 (1966).

2. CROSSING-SYMMETRIC N/D EQUATIONS

In this section, the crossing-symmetric equations of Chew and Mandelstam[2] are rewritten in a form more convenient for the nonlinear analysis of Secs. 3 and 4.

Let the scattering amplitude in a state of isospin I and angular momentum l be $A_l^I(s)$, where s is the usual invariant energy square. Then an N/D decomposition will be assumed in the form

$$A_l^I(s) = (s - 4\mu^2)^l N_l^I(s)/D_l^I(s), \tag{2.1}$$

where μ is the pion mass. One can write down unsubtracted dispersion relations for $N_l^I(s)$ and $D_l^I(s)$:

$$N_l^I(s) = P_l^I(s) + \frac{1}{\pi}\int_{-\Lambda}^{0} \frac{ds'}{s'-s}\,\mathrm{Im}\,A_l^I(s') \times D_l^I(s')(s' - 4\mu^2)^{-l}, \tag{2.2}$$

$$D_l^I(s) = Q_l^I(s) - \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{ds'}{s'-s}\left(\frac{s'-4\mu^2}{s'}\right)^{\frac{1}{2}} \times R_l^I(s')N_l^I(s')(s' - 4\mu^2)^{l}. \tag{2.3}$$

In these equations, a cutoff has been imposed on the left-hand cut, at $s = -\Lambda$. The absorptive part $\mathrm{Im}\,A_l^I(s)$ is to be determined for $-\Lambda < s \leq 0$ by crossing. The inelasticity function $R_l^I(s)$ satisfies

$$\mathrm{Im}\,A_l^I(s) = R_l^I(s)\,|A_l^I(s)|^2 \tag{2.4}$$

and this is supposed given. It would seem at first that the D equation might require subtractions in general, and that the functions $P_l^I(s)$, $Q_l^I(s)$ could be arbitrary holomorphic functions. It will be shown, however, that Eq. (2.2) can be written without subtractions, if one allows $P_l^I(s)$ and $Q_l^I(s)$ to contain poles. For suppose a certain decomposition (2.1) existed in which $N_l^I(s) = O(s^\alpha)$ for $s \to \infty$, with $\alpha > -l$. Then a new decomposition could be defined by the replacements

$$N_l^I(s) \to \frac{N_l^I(s)}{\prod_{r=1}^{n}(s-s_r)}, \qquad D_l^I(s) \to \frac{D_l^I(s)}{\prod_{r=1}^{n}(s-s_r)}, \tag{2.5}$$

where the s_r are arbitrary, real, and distinct points satisfying $0 < s_r < 2\mu^2$, and where n is any integer greater than $\alpha + l$. Thus, the new integral in Eq. (2.3) will converge with no subtractions (assuming that $R_l^I(s)$ is bounded); $P_l^I(s)$ and $Q_l^I(s)$ will be modified by the replacements (2.5). Since the new $N_l^I(s)$ and the integral in Eq. (2.2) tend to zero as $s \to \infty$, it follows that $P_l^I(s)$ can be at most a sum of poles. If $Q_l^I(s) \not\to 0$, as $s \to \infty$, a further replacement of the form (2.5) can be made, including a sufficient number of factors so that the modified $Q_l^I(s)$ does tend to zero, and can thus be at most a sum of poles.

Equations (2.2) and (2.3) are not explicit in one respect: $N_l^I(s)(s - 4\mu^2)^l$ must tend to zero as $s \to \infty$. To avoid the introduction of l nonlinear moment conditions, another transformation will be made. At the same time the variable s will be replaced by the symmetric, dimensionless variable

$$\omega \equiv \frac{s - 4\mu^2}{2\mu^2}. \tag{2.6}$$

The final transformation is

$$n_l^I(\omega) \equiv \prod_{r=1}^{l}(s - \bar{s}_r)N_l^I(s), \tag{2.7}$$

$$d_l^I(\omega) \equiv \frac{\prod_{r=1}^{l}(s-\bar{s}_r)}{(s-4\mu^2)^l}D_l^I(s), \tag{2.8}$$

where, again, the $\bar{s}_r$ are arbitrary points in $(0, 2\mu^2)$.

It follows from Eq. (2.1) that

$$A_l^I(s) = n_l^I(\omega)/d_l^I(\omega) \tag{2.9}$$

and the new equations are

$$n_l^I(\omega) = p_l^I(\omega) + \frac{1}{\pi}\int_{-\lambda}^{-1}\frac{d\omega'}{\omega' - \omega}\,\alpha_l^I(-\omega')\,d_l^I(\omega'), \tag{2.10}$$

$$d_l^I(\omega) = q_l^I(\omega) + \frac{1}{(\omega-1)^l} - \frac{1}{\pi}\int_1^{\infty}\frac{d\omega'}{\omega'-\omega} \times \rho(\omega')r_l^I(\omega')n_l^I(\omega'), \tag{2.11}$$

where

$$\begin{aligned} \rho(\omega) &\equiv [(\omega-1)/(\omega+1)]^{\frac{1}{2}}, \\ r_l^I(\omega) &\equiv R_l^I(s), \\ \lambda &\equiv 1 + \Lambda/2\mu^2, \\ \alpha_l^I(-\omega) &\equiv \operatorname{Im} A_l^I(s), \quad s < 0. \end{aligned} \tag{2.12}$$

The threshold condition has been made automatic by the introduction of the lth order pole in the d equation. A more general form of the d equation would contain a term $t_{l-1}^I(\omega)(\omega - 1)^{-l}$ in place of $(\omega - 1)^{-l}$, where $t_{l-1}^I(\omega)$ is an $(l-1)$th-order polynomial. This polynomial could be removed by dividing $n_l^I(\omega)$ and $d_l^I(\omega)$ by $t_{l-1}^I(\omega)$, which would effect a modification in the forms $p_l^I(\omega)$ and $q_l^I(\omega)$. It has been shown that $p_l^I(\omega)$ and $q_l^I(\omega)$ can be written as sums of poles at arbitrary positions in the interval $(-1, 0)$,

$$p_l^I(\omega) = \sum_{r=1}^{\nu}\frac{R_r}{\omega - \rho_r}, \tag{2.13}$$

$$q_l^I(\omega) = \sum_{r=1}^{\mu}\frac{T_r}{\omega - \tau_r}, \tag{2.14}$$

where μ, ν, R_r, T_r, ρ_r, τ_r can all depend on I and l. In the more general form, $p_l^I(\omega)$ and $q_l^I(\omega)$ would contain poles also at the sites of the zeros of the polynomial $t_{l-1}^I(\omega)$ that was introduced above. If none of these zeros lies in the physical region $(1, \infty)$, there is no difficulty in generalizing the proof; this is left to the interested reader. In this paper, it will be supposed that each of the ρ_r, τ_r lies in the interval $[-1, 0]$. Since $A_l^I(s)$ has no poles (i.e., there are no bound states in the $\pi\pi$ system), then for every pole of $p_l^I(\omega)$ there must occur, at the same position, a pole of $q_l^I(\omega)$. Hence, $q_l^I(\omega)$ has at least as many poles as has $p_l^I(\omega)$. The fact that $p_l^I(\omega)$ and $q_l^I(\omega)$ are arbitrary constitutes the CDD ambiguity. Poles of $q_l^I(\omega)$ that do not correspond to poles of $p_l^I(\omega)$ may be called CDD poles of the first kind ("classical CDD poles"), whereas coincident poles of $p_l^I(\omega)$ and $q_l^I(\omega)$ could be referred to as CDD poles of the second kind. They correspond to simultaneous subtractions of the N and D equations in the formulation in which no poles are allowed in N.

It is necessary that Eqs. (2.10) and (2.11) be modified slightly for the two S-wave amplitudes. In the form of Eq. (2.11), it is not possible to exclude a zero of $d_0^I(\omega)$ at the threshold $\omega = 1$ (i.e., an infinite S-wave scattering length). Since such an eventuality would vitiate the subsequent proof, a simple pole is retained in the d equation at threshold, with a cancelling pole in the n equation. Thus, for the S waves one has

$$n_0^I(\omega) = p_0^I(\omega) + \frac{a^I}{\omega - 1} + \frac{1}{\pi}\int_{-\lambda}^{-1}\frac{d\omega'}{\omega'-\omega}\,\alpha_0^I(\omega')\,d^I(\omega'), \tag{2.15}$$

$$d_0^I(\omega) = q_0^I(\omega) + \frac{1}{\omega - 1} - \frac{1}{\pi}\int_1^{\infty}\frac{d\omega'}{\omega'-\omega}\,\rho(\omega')r_0^I(\omega')n_0^I(\omega'), \tag{2.16}$$

where a^I is the S-wave scattering length for isospin I.

The left-hand cut discontinuity $\alpha_l^I(-\omega)$ is to be determined by crossing symmetry. The usual crossing relation[2] is

$$\operatorname{Im} A_l^I(s) = \frac{1}{s-4\mu^2}\sum_{I'=0}^{2}\beta_{II'}\sum_{\substack{l'=0\\(I'+l')\text{even}}}^{\infty}(2l'+1) \times \int_{4\mu^2}^{4\mu^2-s} ds' P_l\left(1+\frac{2s'}{s-4\mu^2}\right) \times P_{l'}\left(1+\frac{2s}{s'-4\mu^2}\right)\operatorname{Im} A_{l'}^{I'}(s') \qquad (2.17)$$

for $0 \geq s \geq -\Lambda$, where it is known that the infinite series converges only when $s > -9\mu^2$. The isospin crossing matrix is

$$\beta_{II'} = \begin{bmatrix} \frac{2}{3} & 2 & \frac{10}{3} \\ \frac{2}{3} & 1 & -\frac{5}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{bmatrix}. \qquad (2.18)$$

Equation (2.17) is now rewritten in terms of $\alpha_l^I(\omega)$, and the substitution of $-\omega$ for ω is made throughout. It will be convenient to introduce the quantity $\Xi_{l'}^{I'}(\omega')$ by the following definition:

$$\operatorname{Im} A_{l'}^{I'}(s') \equiv \left[\frac{\omega'-1}{2(\lambda-1)}\right]^{2l'+\frac{1}{2}} \Xi_{l'}^{I'}(\omega') \qquad (2.19)$$

for $1 \leq \omega' < \infty$. The factor $(\omega'-1)^{2l'+1/2}$ is the known threshold behavior of $\operatorname{Im} A_{l'}^{I'}(s')$, and the denominator $2(\lambda-1)$ is introduced as a convergence factor, as will become clear later. Actually, it will only be necessary to work with $\Xi_{l'}^{I'}(\omega')$ in the range $1 \leq \omega' \leq \lambda$. Then Eq. (2.17) has the form

$$\alpha_l^I(\omega) = \frac{1}{\omega+1}\sum_{I'=0}^{2}\beta_{II'}\sum_{\substack{l'=0\\(I'+l')\text{even}}}^{L}\int_1^{\omega} d\omega' \times P_l\left(1-2\frac{\omega'+1}{\omega+1}\right)P_{l'}\left(1-2\frac{\omega-1}{\omega'-1}\right) \times \left[\frac{(\omega'-1)}{2(\lambda-1)}\right]^{2l'+\frac{1}{2}}\Xi_{l'}^{I'}(\omega')(2l'+1), \qquad (2.20)$$

where the partial-wave series has been cut off at $l' = L$. It is felt that an attempt to extend the proof to $L = \infty$ would probably be fruitless within the framework of the N/D equations, since the rate of convergence of the Legendre series depends on the positions of the boundaries of the double-spectral functions.[2] This information is not contained in the N/D equations. Accordingly, any attempt to encompass an infinite number of crossed waves would have to employ the analyticity in $s \otimes t$, and would probably progress the more simply by not taking recourse to partial-wave projections at all. Such considerations lie outside the scope of the present work. It will be shown that, under certain conditions, solutions exist for arbitrary, but finite λ and L. However, one would expect that any physically sensible solution, if such exists, would have $\lambda \leqslant \frac{11}{2}$ (i.e., $\Lambda \leqslant 9\mu^2$) and L not too large.

Lastly, since the absorptive part on the right-hand cut can be expressed in terms of N and D, one has

$$\Xi_l^I(\omega) = \left[\frac{2(\lambda-1)}{\omega-1}\right]^{2l+\frac{1}{2}} \times \frac{\rho(\omega)r_l^I(\omega)[n_l^I(\omega)]^2}{[E_l^I(\omega)]^2+[\rho(\omega)r_l^I(\omega)n_l^I(\omega)]^2} \qquad (2.21)$$

for $1 \leq \omega \leq \lambda$, where

$$E_l^I(\omega) = \operatorname{Re} d_l^I(\omega), \quad \omega \geq 1. \qquad (2.22)$$

The object is to study the nonlinear system of Eqs. (2.10), (2.11), (2.15), (2.16), (2.20), and (2.21) for $n_l^I(\omega)$ and $d_l^I(\omega)$. Of these equations, only (2.10) and (2.16) involve negative values of ω. This disparity can be removed for $l \geq 1$ by substituting Eq. (2.11) into Eq. (2.12), thus obtaining the following integral equation for $n_l^I(\omega)$:

$$n_l^I(\omega) = p_l^I(\omega) + v_l^I(\omega) + h_l^I(\omega) + \frac{1}{\pi}\int_1^{\infty} d\omega' \frac{F_l^I(\omega')-F_l^I(\omega)}{\omega'-\omega} \times \rho(\omega')r_l^I(\omega')n_l^I(\omega'), \qquad (2.23)$$

where

$$F_l^I(\omega) = \frac{1}{\pi}\int_1^{\lambda}\frac{d\omega'}{\omega'+\omega}\alpha_l^I(\omega'), \qquad (2.24)$$

$$h_l^I(\omega) = \frac{(-1)^l}{\pi}\int_1^{\lambda}\frac{d\omega'}{\omega'+\omega}\frac{\alpha_l^I(\omega')}{(\omega'+1)^l}, \qquad (2.25)$$

$$v_l^I(\omega) = \frac{1}{\pi}\int_1^{\lambda}\frac{d\omega'}{\omega'+\omega}\alpha_l^I(\omega')\sum_{r=1}^{\mu}\frac{T_r}{\omega'-\tau_r}. \qquad (2.26)$$

The corresponding equation for the S waves is

$$n_0^I(\omega) = \frac{a^I}{\omega-1} + p_0^I(\omega) + v_0^I(\omega) + h_1^I(\omega) + \frac{1}{\pi}\int_1^{\infty} d\omega' \frac{F_0^I(\omega')-F_0^I(\omega)}{\omega'-\omega} \times \rho(\omega')r_0^I(\omega')n_0^I(\omega'). \qquad (2.27)$$

Finally, the equation for $E_l^I(\omega)$, the real part of $d_l^I(\omega)$, is obtained by taking the principal value of the integral in Eq. (2.11) for $l \geq 1$, and in Eq. (2.16) for $l = 0$.

3. BOUNDEDNESS OF THE N/D EQUATIONS

The Eqs. (2.11), (2.16), (2.20), (2.21), (2.23), and (2.27) can be construed as one nonlinear equation for $\Xi_l^I(\omega)$ in terms of itself:

$$\Xi_l^I(\omega) = \theta \Xi_l^I(\omega), \tag{3.1}$$

where θ is a nonlinear operator that summarizes the equations cited above.

One can define the following norm for any bounded, real function $F(\omega)$:

$$\|F(\omega)\| \equiv \max_{1 \le \omega < \infty} |F(\omega)|. \tag{3.2}$$

All the functions possessing such a norm form a Banach space, and it will be the object of the ensuing proof to show that there exists a positive number Z such that, under certain conditions to be precised, a solution of Eq. (3.1) exists, satisfying

$$\|\Xi_l^I(\omega)\| \le Z^2. \tag{3.3}$$

Since $\Xi_l^I(\omega)$ is only used in the interval $1 \le \omega \le \lambda$ (below the cutoff), it is convenient to set it identically to zero for $\lambda < \omega < \infty$.

The demonstration will be based on the Schauder fixed-point principle. Let $\Lambda_l^I(\omega)$ be defined by

$$\Lambda_l^I(\omega) \equiv \theta \Xi_l^I(\omega). \tag{3.4}$$

It is shown in this section that a Z exists, such that, if $\Xi_l^I(\omega)$ is *any* function satisfying Eq. (3.3), then

$$\|\Lambda_l^I(\omega)\| \le Z^2. \tag{3.5}$$

In Sec. 4, it will be shown that θ is a continuous operator, and that the set of all functions $\Lambda_l^I(\omega)$, defined by Eq. (3.4), is compact. Then, according to the Schauder principle, Eq. (3.4) has at least one fixed point, that is to say, Eq. (3.1) has at least one solution, and it satisfies Eq. (3.3). This is explained more fully in Appendix A, where the reader is also reminded of the meanings of some of these terms.

A. Boundedness of α

The first part of the proof is to show that a Z exists such that Eq. (3.3) implies Eq. (3.5). The first step in this program is to obtain a bound for $\alpha_l^I(\omega)$, given by Eq. (2.20), assuming that Eq. (3.3) holds for some Z. Hence, one has

$$|\alpha_l^I(\omega)| \le \frac{Z^2}{\omega+1} \sum_{I'=0}^{2} |\beta_{II'}| \sum_{\substack{l'=0 \\ (I'+l')\text{even}}}^{L} \int_1^{\omega} d\omega' \, (2l'+1) \times P_{l'}\left(2\frac{\omega-1}{\omega'-1} - 1\right) \left[\frac{\omega'-1}{2(\lambda-1)}\right]^{2l'+\frac{1}{2}}. \tag{3.6}$$

To obtain this result, observe that $1 \le \omega' \le \omega$ implies

$$-1 \le 1 - 2\frac{\omega'+1}{\omega+1} \le 1 - \frac{4}{\omega+1}$$

and

$$-\infty \le 1 - 2\frac{\omega-1}{\omega'-1} \le -1,$$

so that

$$\left|P_l\left(1 - 2\frac{\omega'+1}{\omega+1}\right)\right| \le 1$$

and

$$\left|P_{l'}\left(1 - 2\frac{\omega-1}{\omega'-1}\right)\right| = P_{l'}\left(2\frac{\omega-1}{\omega'-1} - 1\right),$$

which is positive definite.

It is then easy to show that, for $1 \le \omega \le \lambda$,

$$|\alpha_l^I(\omega)| < 12Z^2 \int_1^{\infty} \frac{dx}{(x+1)^{\frac{5}{2}}} \sum_{l'=0}^{\infty} \frac{P_{l'}(x)}{(x+1)^{2l'}} (2l'+1), \tag{3.7}$$

where several trivial maximizations have been performed, and where the upper limit of the partial-wave series L has been replaced by ∞. The series and integral can be performed analytically (Appendix C), and the result gives

$$|\alpha_l^I(\omega)| < 8Z^2 \tag{3.8}$$

for $1 \le \omega \le \lambda$.

B. Boundedness of F, h, and v

From the definitions (2.24) and (3.8), one finds

$$|F_l^I(\omega)| < \frac{8(\lambda-1)Z^2}{\pi\omega} \tag{3.9}$$

for any $1 \le \omega < \infty$. Not only does Eq. (3.8) ensure that $F_l^I(\omega)$ is bounded, it also guarantees its (Hölder) smoothness. Specifically,

$$\frac{F_l^I(\omega') - F_l^I(\omega)}{\omega' - \omega} = \frac{1}{\pi} \int_1^{\lambda} \frac{d\omega''}{(\omega''+\omega)(\omega''+\omega')} \alpha_l^I(\omega''), \tag{3.10}$$

so that

$$\left|\frac{F_l^I(\omega) - F_l^I(\omega')}{\omega - \omega'}\right| < \frac{8(\lambda-1)Z^2}{\pi\omega\omega'}. \tag{3.11}$$

Similarly, Eq. (2.25) yields

$$|h_l^I(\omega)| < \frac{8(\lambda-1)Z^2}{\pi\omega}\frac{1}{2}, \tag{3.12}$$

since $l \ge 1$, and (2.26) gives

$$|v_l^I(\omega)| < \frac{8(\lambda-1)Z^2}{\pi\omega} q, \tag{3.13}$$

where

$$q = \max_{\substack{(I,l) \\ l \ne 0}} \left\| \sum_{r=1}^{\mu} \frac{T_r}{\omega - \tau_r} \right\| \le \max_{\substack{(I,l) \\ l \ne 0}} \sum_{r=1}^{\mu} |T_r| \tag{3.14}$$

(remember that the CDD pole positions and residues are functions of I, l, and that $\tau_r < 0$).

C. Boundedness of n

For the cases $l \geq 1$, the integral Eq. (2.23) is Fredholm. One can assert the existence and square integrability of $n_l^I(\omega)$ if unity is not an eigenvalue of the kernel

$$\frac{1}{\pi}\frac{F_l^I(\omega') - F_l^I(\omega)}{\omega' - \omega}\rho(\omega')r_l^I(\omega').$$

From Eq. (3.11) one sees that

$$\iint_1^\infty d\omega\, d\omega' \left|\frac{1}{\pi}\frac{F_l^I(\omega') - F_l^I(\omega)}{\omega' - \omega}\rho(\omega')r_l^I(\omega')\right|^2 < \frac{64(\lambda - 1)^2 r^2 Z^4}{\pi^4}, \quad (3.15)$$

where

$$r = \max_{I,l} \|r_l^I(\omega)\|. \quad (3.16)$$

The left-hand side of Eq. (3.15) is an upper bound for $1/\lambda_0^2$, where λ_0 is the smallest eigenvalue of the kernel. Hence, if one requires

$$Z^2 < \frac{\pi^2}{8(\lambda - 1)r}, \quad (3.17)$$

the existence and square integrability of $n_l^I(\omega)$ are assured. Equation (2.23) implies that

$$\begin{aligned}|n_l^I(\omega) - p_l^I(\omega) - v_l^I(\omega) - h_l^I(\omega)|^2 &\leq \int_1^\infty d\omega'\left[\frac{1}{\pi}\frac{F_l^I(\omega') - F_l^I(\omega)}{\omega' - \omega}\rho(\omega')r_l^I(\omega')\right]^2 \times \int_1^\infty d\omega'[n_l^I(\omega')]^2 \\ &\leq \frac{64(\lambda - 1)^2 r^2 Z^4}{\pi^4}\frac{1}{\omega^2}\int_1^\infty d\omega'[n_l^I(\omega')]^2, \quad (3.18)\end{aligned}$$

where Schwartz's inequality has been used. Since the integral on the right-hand side of Eq. (3.18) exists, the inequality implies that $\omega n_{l,n}^I(\omega)$ is bounded. In fact,

$$|\omega n_l^I(\omega)| \leq |\omega p_l^I(\omega)| + |\omega v_l^I(\omega)| + |\omega h_l^I(\omega)| + x\,\|\omega n_l^I(\omega)\|, \quad (3.19)$$

where

$$x \equiv \frac{8(\lambda - 1)r}{\pi^2}Z^2 < 1 \quad (3.20)$$

thus,

$$\|\omega n_l^I(\omega)\| \leq \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x}, \quad (3.21)$$

where

$$p = \max_{\substack{I,l\\ l\neq 0}}\left\|\omega\sum_{r=1}^{\nu}\frac{R_r}{\omega - \rho_r}\right\| \leq \max_{\substack{I,l\\ l\neq 0}}\sum_{r=1}^{\nu}|R_r|. \quad (3.22)$$

For the S wave, the relevant integral equation is (2.27). This is not Fredholm, since the inhomogeneous term $a^I/(\omega - 1)$ is not square integrable. This difficulty can be circumvented by defining

$$\varphi_0^I(\omega) \equiv \left(\frac{\omega - 1}{\omega + 1}\right)^{\frac{3}{4}} n_0^I(\omega) \quad (3.23)$$

for then Eq. (2.27) is transformed into the following *Fredholm* equation:

$$\varphi_0^I(\omega) = f_0^I(\omega) + \int_0^\infty K_0^I(\omega, \omega')\, d\omega'\varphi_0^I(\omega'), \quad (3.24)$$

where

$$f_0^I(\omega) = \left(\frac{\omega - 1}{\omega + 1}\right)^{\frac{3}{4}} \times \left[\frac{a^I}{\omega - 1} + p_0^I(\omega) + v_0^I(\omega) + h_1^I(\omega)\right], \quad (3.25)$$

and

$$K_0^I(\omega, \omega') = \frac{1}{\pi}\left(\frac{\omega - 1}{\omega + 1}\right)^{\frac{3}{4}}\frac{F_0^I(\omega') - F_0^I(\omega)}{\omega' - \omega} \times \left(\frac{\omega' + 1}{\omega' - 1}\right)^{\frac{1}{4}} r_0^I(\omega'). \quad (3.26)$$

Now,

$$\begin{aligned}\iint_1^\infty d\omega\, d\omega' K_0^I(\omega, \omega')^2 &< x^2\int_1^\infty \frac{d\omega}{\omega^2}\left(\frac{\omega - 1}{\omega + 1}\right)^{\frac{3}{2}}\int_1^\infty \frac{d\omega'}{\omega'^2}\left(\frac{\omega' + 1}{\omega' - 1}\right)^{\frac{1}{2}} \\ &< 4x^2; \quad (3.27)\end{aligned}$$

it will actually be convenient to require

$$x < 2^{-\frac{3}{2}} \quad (3.28)$$

(see Eq. (3.32) below). Then one can certainly assert the existence and square integrability of $\varphi_0^I(\omega)$. Condition (3.28) is of course stronger than (3.20).

As before, Schwartz's inequality gives

$$\begin{aligned}|\varphi_0^I(\omega) - f_0^I(\omega)|^2 &\leq \int_1^\infty |K_1^I(\omega, \omega')|^2\, d\omega'\int_1^\infty |\varphi_0^I(\omega')|^2\, d\omega' \\ &\leq \frac{4x^2}{\omega^2}\int_1^\infty |\varphi_0^I(\omega')|^2\, d\omega'. \quad (3.29)\end{aligned}$$

In this case $\|\varphi_0^I(\omega)\|$ does not exist, because $f_0^I(\omega)$ diverges at $\omega = 1$. However, the quantity

$$(\omega - 1)n_0^I(\omega) = (\omega - 1)^{\frac{1}{4}}(\omega + 1)^{\frac{3}{4}}\varphi_0^I(\omega) \quad (3.30)$$

is bounded. In fact,

$$\begin{aligned}|(\omega - 1)n_0^I(\omega)| &\leq |a^I| + |(\omega - 1)p_0^I(\omega)| + |(\omega - 1)v_0^I(\omega)| \\ &\quad + |(\omega - 1)h_1^I(\omega)| + 2x\frac{(\omega - 1)^{\frac{1}{4}}(\omega + 1)^{\frac{3}{4}}}{\omega} \\ &\quad \times \|(\omega - 1)n_0^I(\omega)\|\left[\int_1^\infty \frac{d\omega'}{(\omega' - 1)^{\frac{1}{2}}(\omega' + 1)^{\frac{3}{2}}}\right]^{\frac{1}{2}} \\ &\leq a + p_0 + \frac{\pi}{r}(q_0 + \tfrac{1}{2})x + 2^{\frac{3}{4}}x\,\|(\omega - 1)n_0^I(\omega)\|, \quad (3.31)\end{aligned}$$

where

$$p_0 = \max_{I=0,2} \left[\sum_{r=1}^{\nu} |R_r| \right]_{l=0},$$

$$q_0 = \max_{I=0,2} \left[\sum_{r=1}^{\nu} |T_r| \right]_{l=0},$$

and

$$a = \max_{I=0,2} |a^I|.$$

Thus,

$$\|(\omega - 1)n_0^I(\omega)\| \leq \frac{a + p_0 + (\pi/r)(q_0 + \frac{1}{2})x}{1 - 2^{\frac{3}{2}}x}. \quad (3.32)$$

D. Lower Bound for E

A difficulty involved in finding a bound for E is that it is defined by means of a principal-value integral. Thus, it is not enough for $\omega n_l^I(\omega)$ [or $(\omega - 1)n_0^I(\omega)$] to be bounded, one must also have boundedness of the derivative of $n_l^I(\omega)$. However, a bound for this derivative can be obtained, since $n_l^I(\omega)$ satisfies the integral Eq. (2.23) (the cases $l \geq 1$ are treated first), and the boundedness of its first derivative is guaranteed by that of the second derivative of $F_l^I(\omega)$. The latter can be deduced from the form of Eq. (2.24).

In order to calculate the bound on $E_l^I(\omega)$, it will be necessary first to remove the principal value integral. One has

$$\begin{aligned} E_l^I(\omega) &= \frac{1}{(\omega - 1)^l} + q_l^I(\omega) \\ &\quad - \frac{1}{\pi}\int_1^\infty \frac{d\omega'}{\omega'} r_l^I(\omega')\rho(\omega')n_l^I(\omega') \\ &\quad - \frac{\omega}{\pi}\int_1^\infty \frac{d\omega'}{\omega'} r_l^I(\omega')\rho(\omega') \frac{n_l^I(\omega') - n_l^I(\omega)}{\omega' - \omega} \\ &\quad - \omega n_l^I(\omega) \frac{P}{\pi}\int_1^\infty \frac{d\omega' r_l^I(\omega')\rho(\omega')}{\omega'(\omega' - \omega)} \end{aligned} \quad (3.33)$$

which is algebraically equivalent to the real part of Eq. (2.11). However, here the only singular integral involves the known function $r_l^I(\omega)$. It is supposed that $r_l^I(\omega)$ is sufficiently smooth that the following quantity exists:

$$B \equiv \max_{I,l} \left\| \frac{P}{\pi}\int_1^\infty \frac{d\omega' r_l^I(\omega')\rho(\omega')}{\omega'(\omega' - \omega)} \right\|. \quad (3.34)$$

To obtain a bound on the second integral in Eq. (3.33) one returns to Eq. (2.23), which implies

$$\left| \frac{n_l^I(\omega) - n_l^I(\omega')}{\omega - \omega'} \right| \leq \frac{p + (\pi/r)(q + \frac{1}{2})x}{\omega\omega'} + \frac{1}{\pi}\int_1^\infty d\omega'' \frac{\pi x}{r} \times \frac{1}{\omega\omega'\omega''} \rho(\omega'')r_l^I(\omega'') \frac{\|\omega n_l^I(\omega)\|}{\omega''}, \quad (3.35)$$

where several trivial maximizations have been performed, for example,

$$\begin{aligned} &\left| \frac{1}{\omega - \omega'} \left[\frac{F_l^I(\omega'') - F_l^I(\omega)}{\omega'' - \omega} - \frac{F_l^I(\omega'') - F_l^I(\omega')}{\omega'' - \omega'} \right] \right| \\ &= \left| \frac{1}{\pi}\int_1^\lambda \frac{d\omega'''}{(\omega''' + \omega)(\omega''' + \omega')(\omega''' + \omega'')} \alpha_l^I(\omega''') \right| \\ &< \frac{\pi x}{r} \frac{1}{\omega\omega'\omega''} \end{aligned} \quad (3.36)$$

when Eq. (3.21) is combined with Eq. (3.36), one finds

$$\left| \frac{n_l^I(\omega) - n_l^I(\omega')}{\omega - \omega'} \right| < \frac{1}{\omega\omega'} \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x}. \quad (3.37)$$

Inequality (3.37) can now be used in conjunction with Eq. (3.33) to give, for all $1 \leq \omega < \infty$,

$$\left| E_l^I(\omega) - \frac{1}{(\omega - 1)^l} \right| < q + \left(\frac{2r}{\pi} + B\right) \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x}. \quad (3.38)$$

The crucial point about this inequality is that the right-hand side is constant, whereas $1/(\omega - 1)^l$ tends to infinity as $\omega \to 1$. Thus, $|(\omega - 1)^l E_l^I(\omega)|$ has a *lower* bound for some finite region including threshold, that is to say the equations have been written in such a way that an infinite scattering length is excluded (this is necessary for the subsequent proof). Clearly such an eventuality would not be excluded for the S wave, if the same inequality (3.38) were used for it, unless an upper bound were set on the CDD pole residues in the $l = 0$ equation, or unless a sign restriction were imposed upon these residues. This is a possible *modus probandi*. However, it necessarily excludes, at the outset, the possibility of an S-wave resonance, and in the present paper the alternative procedure, summarized by Eqs. (2.16) and (2.27), is adopted.

For $l \geq 1$, Eq. (3.38) implies

$$|(\omega - 1)^l E_l^I(\omega)| > 1 - (\omega - 1)^l \zeta \qquad \text{for all} \quad 1 \leq \omega \leq \tilde{\omega}_l, \quad (3.39)$$

where

$$\zeta \equiv \left(\frac{2r}{\pi} + B\right) \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x} + q, \quad (3.40)$$

and $(\tilde{\omega}_l - 1)^l$ is any number less than $1/\zeta$. One can choose

$$\tilde{\omega}_l - 1 = [(\nu - 1)/\nu\zeta)]^{1/l}, \quad (3.41)$$

where ν is a number larger than unity that will be specified later. Thus, for $1 \leq \omega \leq \tilde{\omega}_l$,

$$|(\omega - 1)^l E_l^I(\omega)| > 1 - (\tilde{\omega}_l - 1)^l \zeta = \frac{1}{\nu}. \quad (3.42)$$

This inequality will be of great importance in the application of the fixed-point principle.

For the S wave, there is the difficulty that $n_0^I(\omega)$ is not necessarily bounded at threshold. Hence, instead of the quantity (3.37), one uses

$$\left|\frac{(\omega - 1)n_0^I(\omega) - (\omega' - 1)n_0^I(\omega')}{\omega - \omega'}\right| < \left\{2p_0 + \frac{\pi x}{2^{\frac{1}{2}} r} \frac{r(a + p_0) + [2^{\frac{3}{2}} - (8 - \pi)x](q_0 + \frac{1}{2})}{1 - 2^{\frac{3}{2}} x}\right\} \frac{1}{\omega}; \quad (3.43)$$

this inequality follows from Eq. (2.27) much as Eq. (3.37) follows from Eq. (2.23). Then the equation for $E_0^I(\omega)$ is written

$$E_0^I(\omega) = \frac{1}{\omega - 1} + q_0^I(\omega) - \frac{1}{\pi} \int_1^\infty \frac{d\omega'}{\omega'} r_0^I(\omega') \times \frac{(\omega' - 1)n_0^I(\omega')}{(\omega'^2 - 1)^{\frac{1}{2}}} - \frac{\omega}{\pi} \int_1^\infty d\omega' \frac{r_0^I(\omega')}{\omega'(\omega'^2 - 1)^{\frac{1}{2}}} \times \frac{(\omega' - 1)n_0^I(\omega') - (\omega - 1)n_0^I(\omega)}{\omega' - \omega} - \omega(\omega - 1)n_0^I(\omega) \frac{P}{\pi} \int_1^\infty d\omega' \times \frac{r_0^I(\omega')}{\omega'(\omega'^2 - 1)^{\frac{1}{2}}} \frac{1}{\omega' - \omega}. \quad (3.44)$$

One can show that

$$\omega \frac{P}{\pi} \int_1^\infty \frac{d\omega'}{(\omega'^2 - 1)^{\frac{1}{2}} \omega'(\omega' - \omega)} < \frac{1 + \sqrt{2}}{2}$$

$$1 < \omega < \infty. \quad (3.45)$$

Accordingly, it is supposed that $r_0^I(\omega')$ is sufficiently smooth that a number B_0 exists, such that

$$B_0 = \max_{I=0,2} \left[\omega \frac{P}{\pi} \int_1^\infty d\omega' \frac{r_0^I(\omega')}{\omega'(\omega'^2 - 1)^{\frac{1}{2}}} \frac{1}{\omega' - \omega}\right]. \quad (3.46)$$

Then one can deduce the following inequality from Eq. (3.44):

$$\left|E_0^I(\omega) - \frac{1}{\omega - 1}\right| < \zeta_0, \quad (3.47)$$

where

$$\zeta_0 = q_0 + r p_0 2^{\frac{1}{2}} + \left\{(a + p_0)\left(B_0 + \frac{r}{2^{\frac{1}{2}}} + \frac{r\pi x}{2}\right) + \pi x(q_0 + \tfrac{1}{2}) \times \left[\frac{B_0}{r} + \frac{3}{2^{\frac{1}{2}}} - \left(4 - \frac{\pi}{2}\right)x\right]\right\} \Big/ (1 - 2^{\frac{3}{2}} x) \quad (3.48)$$

so that

$$|(\omega - 1)E_0^I(\omega)| < 1/\nu_0 \quad (3.49)$$

for all $1 \leq \omega \leq \tilde{\omega}_0$, where

$$\tilde{\omega}_0 - 1 = [(\nu_0 - 1)/(\nu_0 \zeta_0)] \quad (3.50)$$

and ν_0 is any number greater than unity. The development of Eqs. (3.47)–(3.50) is an obvious parallel to that of Eqs. (3.39)–(3.42).

E. Boundedness of Λ

It can now be shown that $\Lambda_l^I(\omega)$, defined by the right-hand side of Eq. (2.21), is bounded. The cases $l \geq 1$ will be treated first. It is clear that a bound exists away from threshold. Specifically,

$$\Lambda_l^I(\omega) = \left[\frac{2(\lambda - 1)}{\omega - 1}\right]^{2l+\frac{1}{2}} \frac{1}{\rho(\omega) r_l^I(\omega)} \times \frac{[\rho(\omega) r_l^I(\omega) n_l^I(\omega)]^2}{[E_l^I(\omega)]^2 + [\rho(\omega) r_l^I(\omega) n_l^I(\omega)]^2} \quad (3.51)$$

so that, for $\tilde{\omega}_l \leq \omega \leq \lambda$,

$$|\Lambda_l^I(\omega)| \leq \left[\frac{2(\lambda - 1)}{\tilde{\omega}_l - 1}\right]^{2l+\frac{1}{2}} \left(\frac{\tilde{\omega}_l + 1}{\tilde{\omega}_l - 1}\right)^{\frac{1}{2}}. \quad (3.52)$$

The quantity $\tilde{\omega}_l$ is defined by Eq. (3.41), and it will be seen later that

$$\tilde{\omega}_l - 1 > 1. \quad (3.53)$$

Hence,

$$|\Lambda_l^I(\omega)| < 3^{\frac{1}{2}}[2(\lambda - 1)]^{2l+\frac{1}{2}}(\tilde{\omega}_l - 1)^{-2l} = 3^{\frac{1}{2}}[2(\lambda - 1)]^{2l+\frac{1}{2}} \times \left\{\frac{\nu}{\nu - 1}\left[\left(\frac{2r}{\pi} + B\right)\frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x} + q\right]\right\}^2 \quad (3.54)$$

for $\tilde{\omega}_l \leq \omega \leq \lambda$, $l = 1, 2, \cdots L$.

A bound can also be obtained in the range $1 \leq \omega \leq \tilde{\omega}_l$ by using the fact that $E_l^I(\omega)$ has no zero in this interval. Equation (3.51) is rewritten

$$\Lambda_l^I(\omega) = \frac{[2(\lambda - 1)]^{2l+\frac{1}{2}}}{(\omega + 1)^{\frac{1}{2}}} \times \frac{r_l^I(\omega)[n_l^I(\omega)]^2}{[(\omega - 1)^l E_l^I(\omega)]^2 + [(\omega - 1)^l \rho(\omega) r_l^I(\omega) n_l^I(\omega)]^2}. \quad (3.55)$$

Hence, with the help of Eqs. (3.21), (3.42), one has

$$|\Lambda_l^I(\omega)| < \frac{r}{2^{\frac{1}{2}}}[2(\lambda - 1)]^{2l+\frac{1}{2}}\left[\nu \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x}\right]^2 \quad (3.56)$$

for $1 \le \omega \le \tilde{\omega}_l$. Thus, Eqs. (3.54) and (3.56) imply that $\Lambda_l^I(\omega)$ is bounded throughout $1 \le \omega \le \lambda$.

For the S wave, Eq. (3.52) becomes

$$|\Lambda_0^I(\omega)| < [2(\lambda-1)]^{\frac{1}{2}}\left(\frac{\tilde{\omega}_0+1}{\tilde{\omega}_0-1}\right)^{\frac{1}{2}}\frac{(\lambda-1)^{\frac{3}{2}}}{(\tilde{\omega}_0-1)^2}$$

$$< 6^{\frac{1}{2}}(\lambda-1)^2\left[\frac{\nu_0}{\nu_0-1}\zeta_0\right]^2 \tag{3.57}$$

for $\tilde{\omega}_0 \le \omega \le \lambda$, where Eq. (3.50) has been used, and it has been assumed that $\tilde{\omega}_0 > 2$ (this condition can be checked later from Eqs. (3.68)–(3.70)).

In the range $1 \le \omega \le \tilde{\omega}_0$, Eq. (3.55) is written (for $l = 0$) in the form

$$\Lambda_0^I(\omega) = \frac{[2(\lambda-1)]^{\frac{1}{2}}}{(\omega+1)^{\frac{1}{2}}} \times \frac{r_0^I(\omega)[(\omega-1)n_0^I(\omega)]^2}{[(\omega-1)E_0^I(\omega)]^2 + [(\omega-1)\rho(\omega)r_0^I(\omega)n_0^I(\omega)]^2}. \tag{3.58}$$

Then, Eqs. (3.32) and (3.49) yield

$$|\Lambda_0^I(\omega)| < r(\lambda-1)^{\frac{1}{2}}\left[\nu_0\frac{a+p_0+(\pi/r)(q_0+\frac{1}{2})x}{1-2^{\frac{3}{2}}x}\right]^2 \tag{3.59}$$

for $1 \le \omega \le \tilde{\omega}_0$.

F. Conditions for a Bounded Mapping

In this subsection, it will be shown that values of Z and ν exist such that, for any $\Xi_l^I(\omega)$ that satisfies Eq. (3.3), the right-hand sides of the inequalities (3.54), (3.56), (3.57), and (3.59) are all less than Z^2. It will follow then that Eq. (3.5) holds.

As usual, the cases $l \ge 1$ are considered first. It will be shown that, for a general $\nu > 1$, a Z exists such that the right-hand side of Eq. (3.56) is less than Z^2. Then it will be shown that ν can be chosen sufficiently large to ensure that the right-hand side of Eq. (3.54) is also less than Z^2.

The first step, then, is to show that a Z exists such that

$$r^{\frac{1}{2}}2^L(\lambda-1)^{L+\frac{1}{4}}\nu\,\frac{p+(\pi/r)(q+\frac{1}{2})x}{1-x} < Z \tag{3.60}$$

where the left-hand side here is the square root of the right-hand side of Eq. (3.56). Upon using the definition (3.20), and rearranging the terms, one finds

$$f(Z) \equiv AZ^3 + BZ^2 - Z + C < 0, \tag{3.61}$$

where

$$A = \frac{8(\lambda-1)r}{\pi^2}, \tag{3.62}$$

$$B = 2^{L+3}(\lambda-1)^{L+\frac{5}{4}}\frac{r^{\frac{1}{2}}}{\pi}(q+\tfrac{1}{2})\nu, \tag{3.63}$$

$$C = 2^L(\lambda-1)^{L+\frac{1}{4}}r^{\frac{1}{2}}\nu p. \tag{3.64}$$

Notice that A, B, C are positive. The question is whether a positive Z exists for which this cubic polynomial is negative.

In Appendix B, it is shown that a sufficient condition for $f(Z)$ to be negative for a range of positive Z values is

$$C < \frac{1}{4(B^2+4A)^{\frac{1}{2}}} \tag{3.65}$$

or, in terms of the coefficients (3.62)–(3.64),

$$p < \frac{1}{2^{2L+5}(\lambda-1)^{2L+\frac{3}{2}}r\nu^2[(q+\frac{1}{2})^2 + 2^{-(2L+1)}(\lambda-1)^{-(2L+\frac{3}{2})}\nu^{-2}]^{\frac{1}{2}}}. \tag{3.66}$$

That is to say, the upper bound on the residues of the poles in the n equation must not be too large. This is the type of condition one might have expected.

It is shown in Appendix B that when Eq. (3.65) is satisfied, then $f(Z)$ is negative for

$$Z = \frac{1}{2(B^2+4A)^{\frac{1}{2}}} \tag{3.67}$$

or, using Eqs. (3.62), (3.63), one finds

$$Z = \pi/2^{L+4}(\lambda-1)^{L+\frac{5}{4}}r^{\frac{1}{2}}\nu[(q+\tfrac{1}{2})^2 + 2^{-(2L+1)}(\lambda-1)^{-(2L+\frac{3}{2})}\nu^{-2}]^{\frac{1}{2}}. \tag{3.68}$$

With this value for Z, and any $\nu > 1$, it has been shown that Eq. (3.3) implies that the right-hand side of Eq. (3.56) is less than Z^2. The next stage is to show that ν can be chosen so that the right-hand side of Eq. (3.54) is also less than Z^2, i.e., to show that

$$6^{\frac{1}{4}}2^L(\lambda-1)^{L+\frac{1}{4}} \times \frac{\nu}{\nu-1}\left[\left(\frac{2r}{\pi}+B\right)\frac{p+(\pi/r)(q+\frac{1}{2})x}{1-x}+q\right] < Z. \tag{3.69}$$

By comparing this with Eq. (3.60), one sees that this is accomplished by the following definition:

$$\nu - 1 \equiv \frac{6^{\frac{1}{4}}}{r^{\frac{1}{2}}}\left[\left(\frac{2r}{\pi}+B\right)+\frac{q}{p}\right]. \tag{3.70}$$

This value of ν is inserted into Eqs. (3.66) and (3.68). Then it has been shown that

$$|\Lambda_l^I(\omega)| < Z^2 \tag{3.71}$$

for $1 \le \omega \le \lambda$, $l = 1, 2, \cdots L$.

It remains to be shown that the S wave satisfies the same bound. The bound (3.59) can be seen to be less than (3.56), term by term, if

$$a + p_0 \le [2(\lambda - 1)]^L p,\qquad q_0 \le [2(\lambda - 1)]^L q, \tag{3.72}$$

and if

$$\nu_0 = \nu \frac{1 - 2^{\frac{3}{2}}x}{1 - x}. \tag{3.73}$$

In a similar way, it can be shown that the bound (3.57) is less than (3.54), if the following inequalities are satisfied:

$$a + p_0 \le 2^L(\lambda - 1)^{L-\frac{3}{2}} \frac{\nu_0 - 1}{\nu - 1} \frac{B + 2r/\pi}{B_0 + r2^{-\frac{1}{2}} + r\pi x/2} p,$$

$$q_0 + \tfrac{1}{2} \le 2^L(\lambda - 1)^{L-\frac{3}{2}} \frac{\nu_0 - 1}{\nu - 1} \times \frac{B + 2r/\pi}{B_0 + 3r2^{-\frac{1}{2}} - (4 - \pi/2)x}(q + \tfrac{1}{2}). \tag{3.74}$$

At this point, it is possible to check the inequality (3.53).

Hence, it has been shown that Z, ν, and ν_0 can be chosen such that, if the conditions (3.66), and the stronger of Eqs. (3.72) and (3.74) are satisfied, then $\|\Lambda_l^I(\omega)\|$ is bounded by Z^2 for all $l = 0, 1, \cdots L$.

4. CONTINUITY AND COMPACTNESS

It has been shown in the previous section that if conditions (3.66), (3.72), (3.74) are satisfied, and if Z is given by Eqs. (3.68) and (3.70), then the operator θ maps the set of functions (3.3) into the set (3.5). In this section, it will be shown that this mapping is continuous, and that the functions $\Lambda_l^I(\omega)$ constitute a compact set.

To demonstrate the continuity, one has to show that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\Xi_l^I(\omega)$, $\hat{\Xi}_l^I(\omega)$, satisfying Eq. (3.3), for which

$$\|\Xi_l^I(\omega) - \hat{\Xi}_l^I(\omega)\| < \delta, \tag{4.1}$$

one has

$$\|\theta\Xi_l^I(\omega) - \theta\hat{\Xi}_l^I(\omega)\| < \epsilon, \tag{4.2}$$

i.e.,

$$\|\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)\| < \epsilon. \tag{4.3}$$

It is therefore enough to show that a number P exists such that

$$\|\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)\| < P\,\|\Xi_l^I(\omega) - \hat{\Xi}_l^I(\omega)\| \tag{4.4}$$

for arbitrary $\Xi_l^I(\omega)$, $\hat{\Xi}_l^I(\omega)$ satisfying Eq. (3.3); for then, given an $\epsilon > 0$, one need only choose $\delta = \epsilon/P$ to ensure that Eq. (4.4) follows from Eq. (4.1). The cases $l \ge 1$ will be treated explicitly.

From Eq. (2.20) one has, following a treatment analogous to that leading to Eq. (3.8),

$$|\alpha_l^I(\omega) - \hat{\alpha}_l^I(\omega)| < 8\,\|\Xi_l^I(\omega) - \hat{\Xi}_l^I(\omega)\|. \tag{4.5}$$

Then, from Eq. (2.23),

$$\begin{aligned}|n_l^I(\omega) - \hat{n}_l^I(\omega)| &\le |v_l^I(\omega) - \hat{v}_l^I(\omega)| + |h_l^I(\omega) - \hat{h}_l^I(\omega)| \\ &+ \frac{1}{\pi}\int_1^\infty d\omega' \rho(\omega') r_l^I(\omega') \left| \frac{F_l^I(\omega') - F_l^I(\omega)}{\omega' - \omega} n_l^I(\omega') - \frac{\hat{F}_l^I(\omega') - \hat{F}_l^I(\omega)}{\omega' - \omega} \hat{n}_l^I(\omega')\right|.\end{aligned} \tag{4.6}$$

On using Eqs. (2.24)–(2.26), one finds

$$|\omega n_l^I(\omega) - \omega\hat{n}_l^I(\omega)| < P_1\,\|\Xi_l^I(\omega) - \hat{\Xi}_l^I(\omega)\|, \tag{4.7}$$

where

$$P_1 \equiv \frac{8(\lambda - 1)}{\pi}\left[q + \frac{1}{2} + \frac{rp/\pi + (q + \frac{1}{2})x}{1 - x}\right](1 - x)^{-1}. \tag{4.8}$$

In a similar way, it follows from Eq. (3.33) after some algebra, that

$$|E_l^I(\omega) - \hat{E}_l^I(\omega)| \le P_2\,\|\Xi_l^I(\omega) - \hat{\Xi}_l^I(\omega)\|, \tag{4.9}$$

where

$$P_2 \equiv \left(\frac{2r}{\pi} + B\right)P_1. \tag{4.10}$$

Lastly, one has

$$[\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)] = \left[\frac{2(\lambda - 1)}{\omega - 1}\right]^{2l+\frac{1}{2}} \rho(\omega) r_l^I(\omega) \times \frac{[n_l^I(\omega)\hat{E}_l^I(\omega)]^2 - [\hat{n}_l^I(\omega)E_l^I(\omega)]^2}{[C_l^I(\omega)\hat{C}_l^I(\omega)]^2}, \tag{4.11}$$

where

$$C_l^I(\omega) \equiv \{[E_l^I(\omega)]^2 + [\rho(\omega) r_l^I(\omega) n_l^I(\omega)]^2\}^{\frac{1}{2}}, \tag{4.12}$$

and similarly for $\hat{C}_l^I(\omega)$. The last factor in Eq. (4.11) can be written

$$\frac{\frac{1}{2}(n\hat{E} + \hat{n}E)[(n + \hat{n})(\hat{E} - E) + (\hat{E} + E)(n - \hat{n})]}{[C\hat{C}]^2},$$

where arguments and indices have been omitted. Use is then made of the inequality

$$|E_l^I(\omega)/C_l^I(\omega)| \le 1, \tag{4.13}$$

so that one has

$$\begin{aligned}|\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)| &\le [2(\lambda - 1)]^{2l+\frac{1}{2}}(\omega + 1)^{\frac{1}{2}} r_l^I(\omega)(\omega - 1)^{-2l} \\ &\frac{1}{2}\left[\frac{n_l^I(\omega)}{C_l^I(\omega)} + \frac{\hat{n}_l^I(\omega)}{\hat{C}_l^I(\omega)}\right]\left\{\frac{n_l^I(\omega) + \hat{n}_l^I(\omega)}{C_l^I(\omega)\hat{C}_l^I(\omega)}|E_l^I(\omega) - \hat{E}_l^I(\omega)| \right. \\ &\left. + \left[\frac{1}{C_l^I(\omega)} + \frac{1}{\hat{C}_l^I(\omega)}\right]|n_l^I(\omega) - \hat{n}_l^I(\omega)|\right\}.\end{aligned} \tag{4.14}$$

In the interval $1 \le \omega \le \tilde{\omega}_l$, one sees that the bound (3.42) implies

$$|(\omega - 1)^l C_l^I(\omega)| > \frac{1}{\nu} \tag{4.15}$$

and similarly for $(\omega - 1)^l \hat{C}_l^I(\omega)$. Hence, in this interval one can absorb the factor $(\omega - 1)^{-2l}$ in Eq. (4.14) into the factors $1/C_l^I(\omega)$ and $1/\hat{C}_l^I(\omega)$, at least two of which occur for each term. Hence,

$$|\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)| < P_3\,|n_l^I(\omega) - \hat{n}_l^I(\omega)| + P_4\,|E_l^I(\omega) - \hat{E}_l^I(\omega)| \tag{4.16}$$

for $1 \le \omega \le \tilde{\omega}_l$, where

$$P_3 = [2(\lambda - 1)]^{2l+\frac{1}{2}} 2^{\frac{3}{2}} r\nu^2 \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x},$$

$$P_4 = \nu(\tilde{\omega}_l - 1)^l \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x} P_3.$$

Equations (4.7), (4.9), and (4.16) imply

$$|\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)| < (P_1 P_4 + P_2 P_3)\,\|\Xi_l^I(\omega) - \hat{\Xi}_l^I(\omega)\| \tag{4.17}$$

for $1 \le \omega \le \tilde{\omega}_l$. This is part of the demonstration of Eq. (4.4). It is necessary now to show a similar bound for the interval $\tilde{\omega}_l \le \omega \le \lambda$.

In the interval $\tilde{\omega}_l \le \omega \le \lambda$, the factor $(\omega - 1)^{-2l}$ in Eq. (4.14) is no longer an incipient source of difficulty, since it is finite. However, the inverses of the quantities $C_l^I(\omega)$, $\hat{C}_l^I(\omega)$ are potential problems, since these inverses can, in general, become unbounded. A zero of $C_l^I(\omega)$ corresponds to a simultaneous zero of $E_l^I(\omega)$ and $n_l^I(\omega)$ [see Eq. (4.12)], that is to say, to an extinct bound state.[11] If such a coincidence occurs in the region $\tilde{\omega}_l \le \omega \le \lambda$, the proof of the continuity of the operator θ breaks down. It will be shown that the CDD poles $p_l^I(\omega)$ and $q_l^I(\omega)$ can be chosen in such a way that no extinct bound states occur in the physical region, so that $|C_l^I(\omega)|$ has a lower bound, whence the continuity of θ can be deduced. There will still be a continuum of possibilities for the CDD parameters. In this section, it is shown that $p_l^I(\omega)$ can be chosen so that $n_l^I(\omega)$ has no zeros in the physical region, so that $|n_l^I(\omega)|$ and, hence, $|C_l^I(\omega)|$ have lower bounds. It will then be possible to complete the existence proof for the amplitude $A_l^I(\omega)$. However, it will be shown in Sec. 5 that this is not sufficiently general to exclude ghosts (i.e., poles of $A_l^I(\omega)$ on the physical sheet) in all cases. In that section it will be shown that the zeros of $E_l^I(\omega)$ and $n_l^I(\omega)$ can be made to alternate in the physical region, in such a way that $C_l^I(\omega)$ always has a lower bound, and that $A_l^I(\omega)$ has no ghosts.

[11] D. Atkinson and M. B. Halpern, Phys. Rev. **149**, 1133 (1966).

For the present, it will be shown how $p_l^I(\omega)$ can be chosen so that $n_l^I(\omega)$ has no zeros in the physical region. This will be of use in the more general treatment of Sec. 5. The integral Eq. (2.23) can be manipulated to give

$$\begin{aligned}|\omega n_l^I(\omega) - \omega p_l^I(\omega)| &< x\left[\frac{\pi}{r}(q + \tfrac{1}{2}) + \frac{\pi}{2^{L+\frac{3}{2}}(\lambda - 1)^{L+\frac{3}{4}} r\nu} x^{\frac{1}{2}}\right] \\ &< \frac{\pi\{1 + x^{\frac{1}{2}}/[2^{L+\frac{1}{2}}(\lambda - 1)^{L+\frac{3}{4}} r\nu(q + \frac{1}{2})]\}}{2^{2L+5}(\lambda - 1)^{2L+\frac{3}{2}}\nu^2(q + \frac{1}{2})r}.\end{aligned} \tag{4.18}$$

The function $\omega p_l^I(\omega)$ is a rational function that can be written, in general, as the quotient of two νth-order polynomials. The roots of the denominator lie in the interval $-1 < \omega < 0$. The roots of the numerator give the zeros of $\omega p_l^I(\omega)$. Hence, if $p_l^I(\omega)$ is chosen such that none of the zeros of $\omega p_l^I(\omega)$ are in the physical region $1 \le \omega < \infty$, then $\omega p_l^I(\omega)$ will have an invariable sign throughout the region, and so $|\omega p_l^I(\omega)|$ will have a nonzero lower bound, say min $|\omega p_l^I(\omega)|$. Since $\nu - 1 > \dfrac{6^{\frac{1}{4}}q}{r^{\frac{1}{2}}p}$ [see Eq. (3.70)], it follows that q/p can be made so large that the right-hand side of Eq. (4.18) is smaller than min $|\omega p_l^I(\omega)|$. Then Eq. (4.18) implies that $|\omega n_l^I(\omega)|$ is always greater than the difference between min $|\omega p_l^I(\omega)|$ and the right-hand side. Let η_l^I be this lower bound for $|\omega n_l^I(\omega)|$ in $1 \le \omega < \infty$, for all I, l. Hence, Eq. (4.12) implies

$$|C_l^I(\omega)| > \rho(\tilde{\omega}_l)\eta_l^I \tag{4.19}$$

for all $\tilde{\omega}_l \le \omega \le \lambda$, and similarly for $\hat{C}_l^I(\omega)$. Hence, Eq. (4.14) gives

$$|\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)| < P_3'\,|n_l^I(\omega) - \hat{n}_l^I(\omega)| + P_4'\,|E_l^I(\omega) - \hat{E}_l^I(\omega)| \tag{4.20}$$

for $\tilde{\omega}_l \le \omega \le \lambda$, where

$$P_3' = \left[\frac{2(\lambda - 1)}{\tilde{\omega}_l - 1}\right]^{2l+\frac{1}{2}} 2r(\tilde{\omega}_l^2 - 1)^{\frac{1}{2}} \times [\rho(\tilde{\omega}_l)\eta_l^I]^{-2} \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x},$$

$$P_4' = [\rho(\tilde{\omega}_l)\eta_l^I]^{-1} \frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x} P_3',$$

and so

$$|\Lambda_l^I(\omega) - \hat{\Lambda}_l^I(\omega)| < (P_1 P_4' + P_2 P_3')\,\|\Xi_l^I(\omega) - \hat{\Xi}_l^I(\omega)\| \tag{4.21}$$

for $\tilde{\omega}_l \le \omega \le \lambda$. The reader who has persevered to this point can be safely left to supply a parallel proof for the S wave.

Inequalities (4.17) and (4.21) are equivalent to Eq. (4.4), if one defines

$$P \equiv \max\,[P_1P_4 + P_2P_3,\, P_1P_4' + P_2P_3']. \quad (4.22)$$

With this definition, the continuity of θ is proven.

A. Compactness

It has been shown that θ maps the set of functions that satisfy Eq. (3.3), say T, into itself. It is the purpose of this section to show that θ actually maps T into a compact subset of itself, say T'. That is, if $\Xi_l^I(\omega)$ belongs to T, one must show that $\Lambda_l^I(\omega)$ belongs to T'. This can be done, according to Arzelá's theorem, by showing that the functions $\Lambda^I(\omega)$ are equi-continuous. These ideas are explained in Appendix A.

To demonstrate the equi-continuity of $\Lambda_l^I(\omega)$, it is enough to show the equi-continuity of $n_l^I(\omega)$ and $(\omega - 1)^l E_l^I(\omega)$, since the denominator in Eq. (3.55), namely $[(\omega - 1)^l C_l^I(\omega)]^2$, has a uniform lower bound in $1 \leq \omega \leq \lambda$. It is easy to show from Eq. (2.23) that $n_l^I(\omega)$ is equi-continuous, since each of the terms on the right-hand side is equi-continuous. For example, one can see that the kernel is equi-continuous as follows: Since $\alpha_l^I(\omega)$ is uniformly bounded for $\Xi_l^I(\omega)$ in T, and since $n_l^I(\omega)$ is uniformly bounded, as the work of Sec. 3 shows, and since the integral in Eq. (2.23) converges uniformly with respect to ω, it follows that this integral is equi-continuous. In a similar way, it can be shown, from Eq. (3.33), that $(\omega - 1)^l E_l^I(\omega)$ is equi-continuous, if it is supposed that $r_l^I(\omega)$ is such that

$$\frac{P}{\pi}\int_1^\infty \frac{d\omega' r_l^I(\omega)\rho(\omega')}{\omega'(\omega' - \omega)}$$

is continuous with respect to ω.

Thus, it has been shown that $\Lambda_l^I(\omega)$ is equi-continuous for all $l \neq 0$; and the reader may easily fabricate a similar proof for the S wave. Hence, θ maps T into a compact subset of itself, and the Schauder fixed-point principle is accordingly applicable to the N/D system. Under the restrictions on $p_l^I(\omega)$ and $q_l^I(\omega)$ which have been specified, one can now assert the existence of a solution. However, in general there will be zeros of $d_l^I(\omega)$ on the physical sheet of ω. The purpose of the next section is to show that, by making somewhat more stringent requirements on the CDD pole parameters, one can ensure that all the zeros of $d_l^I(\omega)$ are on unphysical sheets. This involves a generalization of the requirement that $n_l^I(\omega)$ has a constant sign in $1 \leq \omega < \infty$; but the essential point, that $(\omega - 1)^l C_l^I(\omega)$ has a uniform lower bound, is retained.

5. CONDITIONS FOR THE ABSENCE OF GHOSTS

In the previous sections, it has been shown that the crossing symmetric N/D equations have solutions, if certain conditions are satisfied. The main result of interest is probably the fact that the CDD ambiguity is not resolved by the requirement of crossing symmetry for the absorptive part. However, in general, a solution of an arbitrary N/D system will have ghosts; and so one might suggest that the CDD ambiguity may be resolved, after all, if one takes into account the physical requirement that the D function has no zeros. Indeed, it is the case that a "strong" CDD pole is often associated with a nearby ghost, so the temptation to equate a ghost-free solution with the absence of CDD poles is particularly strong.

It is the purpose of this section to dispel that idea, and to show that some CDD poles are consistent with the requirement of no ghosts. The physical reason for this is rather clear. If the CDD pole is in, or near, the physical region, the associated zero of the D function can be displaced in a complex direction into the unphysical sheet. If the sign of the N function is correct, this zero will correspond to a resonance. This is not merely an academic possibility. The ρ-meson occurs in the $I = 1$, $J = 1$ state, and the f^0-meson in $I = 0$, $J = 2$. Moreover, there is rather strong evidence[12] now for a broad σ resonance in $I = 0$, $J = 0$. Any, or all of these states may be associated with a CDD pole. There is even some circumstantial evidence that this is the case for the ρ-meson, as is discussed in the next section.

There are several different ways in which one could define an N/D system which is free from ghosts. The following demonstration will be a sketch only, since it is not intended to be part of the existence proof, and no attempt will be made to set up general conditions for the absence of ghosts.

Equation (3.43) yields, by a sequence which parallels that leading to Eq. (3.48),

$$\left|E_l^I(\omega) - \frac{1}{(\omega - 1)^l} - q_l^I(\omega)\right| < \left(\frac{2r}{\pi} + B\right)\frac{p + (\pi/r)(q + \frac{1}{2})x}{1 - x} \quad (5.1)$$

for $l \neq 0$. The rational function $(\omega - 1)^{-l} + q_l^I(\omega)$ can be written, in general, as the quotient of two $(\mu + l - 1)$th-order polynomials, which will possess $(\mu + l - 1)$ zeros. For large $q_l^I(\omega)$, this rational function will be larger than the right-hand side of

[12] C. Lovelace, R. M. Heinz, and A. Donnachie, Phys. Letters **22**, 332 (1966).

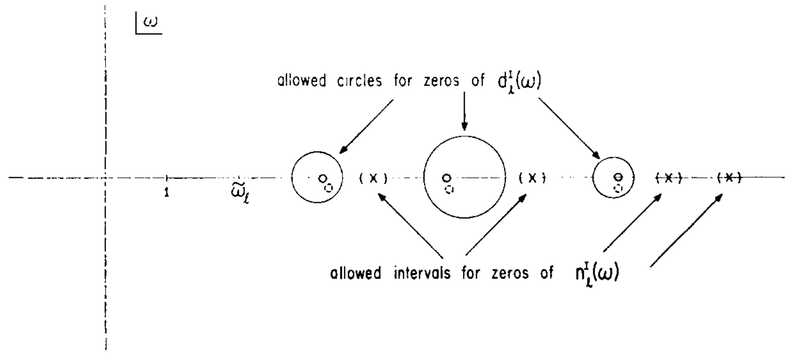


FIG. 1. Interleaving zeros of $n_l^I)\omega)$ and $E_l^I(\omega)$. $\times$ = Zero of $n_l^I(\omega)$, $\bigcirc$ = Zero of $E_l^I(\omega)$, ◌ = Zero of $d_l^I(\omega)$ (on the unphysical sheet).

Eq. (5.1), except in a finite, calculable circle surrounding each of the $(\mu + l - 1)$ zeros mentioned above. Within each circle there will be, somewhere, a zero of $E_l^I(\omega)$, corresponding to the zero of $(\omega - 1)^{-l} + q_l^I(\omega)$ at its center. The object is to show that the zero can be displaced from the center of the circle onto the unphysical sheet. Suppose that $q_l^I(\omega)$ is chosen so that all the zeros of $(\omega - 1)^{-l} + q_l^I(\omega)$ lie along the physical region $1 \leq \omega \leq \infty$. Moreover, suppose that these zeros are sufficiently separated that the circles, centered about each one of them, within which a zero of $E_l^I(\omega)$ lies, do not overlap. Then, $|E_l^I(\omega)|$ has a lower bound outside the union of all these circles.

Suppose that ω_0 is a typical (real) zero of $E_l^I(\omega)$. Then, $d_l^I(\omega)$ can be expanded about $\omega = \omega_0$ to give

$$d_l^I(\omega) \approx -ir_l^I(\omega_0)\rho(\omega_0)n_l^I(\omega_0) + (\omega - \omega_0) \times \{E_l^{I\prime}(\omega_0) - i[r_l^I(\omega_0)\rho(\omega_0)n_l^I(\omega_0)]'\} \quad (5.2)$$

for $\omega \approx \omega_0$. Hence, $d_l^I(\omega) = 0$ for

$$\omega - \omega_0 \approx \frac{ir_l^I(\omega_0)\rho(\omega_0)n_l^I(\omega_0)}{E_l^{I\prime}(\omega_0) - i[r_l^I(\omega_0)\rho(\omega_0)n_l^I(\omega_0)]'}. \quad (5.3)$$

Hence, the imaginary part of this is

$$\operatorname{Im} \omega \approx \frac{r_l^I(\omega_0)\rho(\omega_0)n_l^I(\omega_0)E_l^{I\prime}(\omega_0)}{[E_l^{I\prime}(\omega_0)]^2 + \{[r_l^I(\omega_0)\rho(\omega_0)n_l^I(\omega_0)]'\}^2}. \quad (5.4)$$

If $n_l^I(\omega_0)$ and $E_l^{I\prime}(\omega_0)$ have opposite signs, then $\operatorname{Im} \omega$ is negative, and the zero of $d_l^I(\omega)$ is on the unphysical sheet. Since $E_l^I(\omega)$ has a succession of simple zeros at each of the points ω_0, it follows that the sign of $E_l^{I\prime}(\omega)$ alternates from one zero of $E_l^I(\omega)$ to the next. Hence, it must be shown that $p_l^I(\omega)$, the inhomogeneous term in the n equation, can be chosen in such a way that $n_l^I(\omega)$ alternates in sign between the zeros of $E_l^I(\omega)$, so that $n_l^I(\omega_0)E_l^{I\prime}(\omega_0)$ can be made always negative. It must be shown, therefore, that $n_l^I(\omega)$ can be forced to have real zeros that interleave those of $E_l^I(\omega)$.

It has already been noted that q can be made sufficiently large to ensure that the right-hand side of Eq. (4.22) is smaller than $\omega p_l^I(\omega)$, except in the immediate vicinity of one of the ν zeros of this rational function. Suppose that $\omega p_l^I(\omega)$ is chosen so that its zeros lie in the physical region, in such a way that they interleave the zeros of $(\omega - 1)^{-l} + q_l^I(\omega)$. For this to be possible, it must be supposed that $(\omega - 1)^{-l} + q_l^I(\omega)$, when written out as a rational fraction, has a numerator of order not greater than $(\nu + 1)$, rather than the maximum of $(\mu + l - 1)$. This can certainly be done. While this means that the centrifugal term $(\omega - 1)^{-l}$ imposes a restriction on the CDD pole residues, it is equally clear that it does not determine them. Suppose, more particularly, that the zeros of $\omega p_l^I(\omega)$ are made to lie in those intervals of the real axis where no zeros of $E_l^I(\omega)$ may lie. Around each zero of $\omega p_l^I(\omega)$, one may specify an interval within which a (real) zero of $n_l^I(\omega)$ must lie. By making q large enough, one can ensure that this interval is so small that none of the intervals in which $n_l^I(\omega)$ has a zero overlap any of the intervals in which $E_l^I(\omega)$ has a zero. Hence, the zeros of $n_l^I(\omega)$ and $E_l^I(\omega)$ alternate, and so all the zeros of $d_l^I(\omega)$ can be made to lie on the unphysical sheet. This sketch may be made more transparent by a reference to Fig. 1.

It is clear that the function $C_l^I(\omega)$, defined in Eq. (4.12), has a uniform lower bound throughout $\tilde{\omega}_l \leq \omega < \infty$, since the zeros of $n_l^I(\omega)$ and $E_l^I(\omega)$ lie in mutually exclusive intervals. Hence, the continuity and compactness proofs of Sec. 4 are not disrupted.

6. CONCLUSION

In this paper it has been shown that, under certain conditions, the crossing-symmetric N/D equations have solutions. These conditions may be summarized by saying that the inhomogeneous terms in the amplitude (the N-pole terms divided by the D-pole terms) must not be too large. In the special case that there are *no* inhomogeneities in the N equation, the trivial (identically zero) amplitude satisfies the N/D equations. As the inhomogeneity is perturbed away from zero, so the solution is perturbed away from zero. However, in general the phase shift will not be identically zero in the trivial case. It will be zero, *modulo* π, and will suffer a discontinuity of π at each zero of the D function, if any such occur in the physical region. As the N function discontinuities are increased from zero, these abrupt changes by π are replaced by continuous, resonant phase shifts.

No attempt has been made in this paper to find the largest bounds under which the existence proof will work. In particular, the bound (3.6), in which all the cancellations caused by the oscillations of

$$P_l\left(1 - 2\frac{\omega' + 1}{\omega + 1}\right)$$

in Eq. (2.20) have been ignored, is particularly poor. There can be no doubt that a more detailed account of this cancellation would relax considerably the permitted bounds on the inhomogeneities. Presumably, certain maximal bounds exist, below which the existence of a solution would be provable.

There is already considerable evidence[13,14] that bootstrap models of the ρ-meson do not succeed in producing a sufficiently narrow resonance. On the other hand, an $SU(6)$ model of meson–meson scattering[15] suggests that a one-channel CDD pole should be included in the P-wave of the $\pi\pi$ amplitude. It is clear that such a CDD pole would permit a much narrower ρ resonance (although of course the ρ parameters would only be calculable in terms of the CDD pole residues). On the other hand, it might be that a CDD solution would not permit a sufficiently broad ρ resonance, before a ghost manifested itself in one of the other partial waves, perhaps the $I = 0$ S wave. If, following the work of Lovelace *et al.*,[12] one accepts the existence of a σ-meson, it may be that this state is engendered by the exchange of a CDD ρ-meson. This is, of course, a picturesque, "bootstrap" manner of expression. What is really meant is that perhaps the σ-meson will be produced in a crossing-symmetric $\pi\pi$ system in which there is no CDD pole in the S waves, but one in the P wave. The left-hand cut in the $l = 0$, $I = 0$ equation will be affected by the presence of the CDD pole in the P wave. It may be that an upper limit on the CDD pole strength, and so on the ρ-meson width, would be imposed by the requirement that the σ-meson pole should not encroach upon the physical sheet. It is not clear whether a CDD pole would be necessary in the $J = 2$, $I = 0$ wave, to account for the f^0-meson.

The speculative considerations of the previous paragraph are in part inspired by a series of papers by Shirkov *et al.*,[16,17] who examined CDD branches of the so-called differential approximation of the $\pi\pi$ equations. The general finding is that the ρ width cannot be made large enough, although the σ-meson can be incorporated naturally. It is expected that an existence proof could be constructed for the Shirkov equations, and it is hoped to do this in a future work.

However, it is rather obvious today that a satisfying solution of the $\pi\pi$ problem cannot be given solely in terms of the N/D equations. In particular, it is not clear that the requirement of crossing for the real part of the amplitude would not materially reduce, remove, or even overdetermine the CDD ambiguity. Hence a treatment of the Mandelstam generalized unitarity equations[18] seems imperative, with, presumably, an explicit or implicit projection of the leading complex angular momentum singularities. The difficulties in the way of such an application of the fixed-point theorem are legion, but their mastering would probably yield very interesting information concerning the permitted forms of the "strip cutoff function,"[19] which is a manifestation of the other channels to which the $\pi\pi$ channel is coupled, through unitarity.

ACKNOWLEDGMENTS

Thanks are due to Dr. K. Bardakci, Dr. C. Lovelace, and Dr. J. W. Moffat for interesting discussions.

APPENDIX A

Some fundamental ideas of functional analysis are explained in this Appendix. The definitions and theorems are not given in their most general forms, but only in sufficient generality for the text of this paper.

Let $\Xi(\omega)$ be a function belonging to a Banach space B, and let θ be an operator mapping B into itself. Define

$$\Lambda(\omega) = \theta\Xi(\omega). \tag{A1}$$

Let T be the set of all functions $\Xi(\omega)$ for which

$$\|\Xi(\omega)\| \leq Z^2, \tag{A2}$$

where the left-hand side of Eq. (A2) is the norm of

[13] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965).

[14] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

[15] D. Atkinson and M. B. Halpern, Phys. Rev. **150**, 1377 (1966).

[16] A. V. Efremov, D. V. Shirkov, and H. Y. Tzu, Zh. Eksperim. i Teor. Fiz. **41**, 603 (1961), and Sci. Sinica **10**, 812 (1961).

[17] V. V. Serebriakov and D. V. Shirkov, Zh. Eksperim. i Teor. Fiz. **42**, 610 (1962) [English transl.: Soviet Phys.—JETP **15**, 425 (1962)].

[18] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[19] N. F. Bali, G. F. Chew, and S-Y. Chu, Phys. Rev. **150**, 1352 (1966).

$\Xi(\omega)$, defined in B, and Z is some number. Suppose that θ is such that Eqs. (A1) and (A2) imply

$$\|\Lambda(\omega)\| \leqq Z^2 \tag{A3}$$

for some Z. Then θ is a bounded operator mapping T into some subset of itself, say T'.

The operator θ is said to be continuous for the mapping $T \to T'$, if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|\Xi(\omega) - \hat{\Xi}(\omega)\| < \delta \tag{A4}$$

implies

$$\|\theta\Xi(\omega) - \theta\hat{\Xi}(\omega)\| < \epsilon, \tag{A5}$$

where Ξ and $\hat{\Xi}$ belong to T. Boundedness implies continuity for a linear operator, but not, in general, for a nonlinear operator.

A set in a Banach space, say T', is compact if every infinite sequence of points belonging to T' contains at least one subsequence that converges to some point of T'.

The Fixed-Point Principle of Schauder

If θ maps T *continuously* into a *compact* subset T' of T, then at least one "point" Ξ of T exists for which

$$\Xi(\omega) = \theta\Xi(\omega). \tag{A6}$$

Sections 3 and 4 are concerned with the application of this theorem to the crossing-symmetric N/D equations. The norm that defines the space B is given in Eq. (3.2). In Sec. 3, it is shown that a set T exists such that, if θ maps T into T', then T' is contained in T. In Sec. 4, it is first shown that the operator $\theta(T \to T')$ is continuous. Then it is shown that T' is compact, so that the Schauder principle applies. Hence, a solution of the crossing-symmetric equation (A6) exists.

The proof of continuity is a direct application of the definition Eqs. (A4)–(A5). The compactness of T' is shown by using the following principle:

Arzelà's Theorem: The set T' of functions $\Lambda(\omega)$, defined on $1 \leq \omega \leq \lambda$, is compact if the functions $\Lambda(\omega)$ are uniformly bounded and equi-continuous in $1 \leq \omega \leq \lambda$.

The uniform boundedness follows from the work of Sec. 3. The functions $\Lambda(\omega)$ are said to be equi-continuous if, for any $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|\Lambda(\omega_1) - \Lambda(\omega_2)| < \epsilon \tag{A7}$$

for all ω_1, ω_2 in the closed interval $[1, \lambda]$, for which

$$|\omega_1 - \omega_2| < \delta \tag{A8}$$

and for *all* $\Lambda(\omega)$ in T'. It is this specification of uniformity with respect to Λ that constitutes the difference between continuity and equi-continuity.

APPENDIX B

Suppose that the coefficients A, B, C, are positive, then it is required to find a sufficient condition such that the polynomial

$$f(Z) \equiv AZ^3 + BZ^2 - Z + C \tag{B1}$$

has two real, positive roots.

Consider the quadratic

$$g(Z) \equiv AZ^2 + BZ - 1. \tag{B2}$$

This has one positive root at $Z = Z_1$, given by

$$Z_1 = \frac{1}{2A}[(B^2 + 4A)^{\frac{1}{2}} - B] = \frac{2}{(B^2 + 4A)^{\frac{1}{2}} + B}. \tag{B3}$$

Moreover, $g(0) = -1$. It follows that $Zg(Z)$ is negative between $Z = 0$ and $Z = Z_1$ (see Fig. 2). Hence, $f(Z)$ will have two real, positive roots if C is less than the maximum value of $|Zg(Z)|$. This condition will be expressed in a strengthened form, for simplicity.

Let

$$Z_1' = \frac{1}{(B^2 + 4A)^{\frac{1}{2}}} < Z_1. \tag{B4}$$

Since the chord $(0, -1)$ to $(Z_1, 0)$ lies wholly inside the parabola, it follows *a fortiori* that the line segment $(0, -1)$ to $(Z_1', 0)$ lies inside it (see Fig. 2). The midpoint of this segment is $(Z_0, -\frac{1}{2})$, where

$$Z_0 = \tfrac{1}{2}Z_1' = \frac{1}{2(B^2 + 4A)^{\frac{1}{2}}}. \tag{B5}$$

Hence,

$$g(Z_0) < -\tfrac{1}{2}, \tag{B6}$$

and

$$f(Z_0) < C - \frac{1}{4(B^2 + 4A)^{\frac{1}{2}}}. \tag{B7}$$

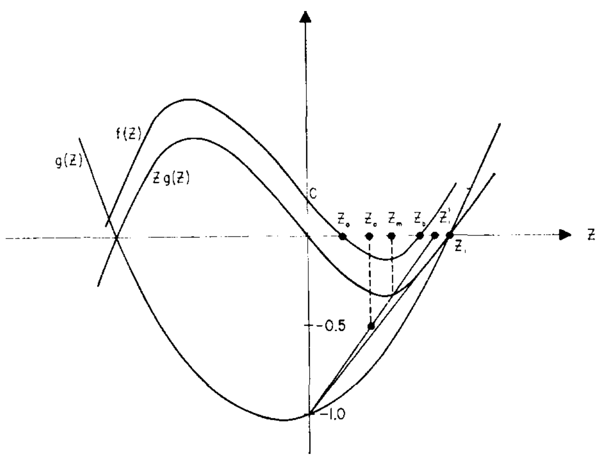


FIG. 2. Cubic polynomial for the N/D system.

Hence, a sufficient condition for two positive roots is

$$C < \frac{1}{4(B^2 + 4A)^{\frac{1}{2}}} \,. \tag{B8}$$

When this condition is satisfied, $f(Z_0)$ will be negative.

APPENDIX C

The generating series for the Legendre polynomials

$$\sum_{l=0}^{\infty} \omega^l P_l(x) = (1 - 2x\omega + \omega^2)^{-\frac{1}{2}} \tag{C1}$$

certainly converges absolutely and uniformly with respect to x, if x and ω are real numbers satisfying

$$x \geq 1,$$

$$0 \leq \omega < \frac{1}{x + (x^2 + 1)^{\frac{1}{2}}} \,. \tag{C2}$$

Hence,

$$\sum_{l=0}^{\infty} (2l + 1) \int_1^{\infty} dx (x + 1)^{-(2l+\frac{5}{2})} P_l(x) = \int_1^{\infty} \frac{[(x + 1)^4 - 1]\, dx}{(x + 1)^{\frac{1}{2}}[(x + 1)^2(x^2 + 1) + 1]^{\frac{3}{2}}} \,, \tag{C3}$$

where the interchange in order of sum and integral is permissible. It can be shown that this integral is less than $\frac{2}{3}$. Hence,

$$\sum_{l=0}^{L} (2l + 1) \int_1^{\infty} dx (x + 1)^{-(2l+\frac{5}{2})} P_l(x) < \tfrac{2}{3} \tag{C4}$$

for any L.

Geometrization of a Complex Scalar Field. II. Analysis

R. PENNEY
Scientific Laboratory, Ford Motor Company, Dearborn, Michigan

(Received 20 April 1967)

The necessary and sufficient conditions that a Ricci tensor should represent the energy tensor of a complex scalar field are given for the general non-null case. The complexion gradient of the field is determined only to within a sign by the Ricci tensor, unlike the case of the electromagnetic problem. The physical content of a field which corresponds to massless "pions" is expressed entirely in geometric terms within the Rainich scheme.

I. INTRODUCTION

In a previous paper,[1] we have found the necessary and sufficient conditions that a Ricci tensor should algebraically represent the energy tensor of a complex scalar field. In the present analysis, we want to solve the problem of specifying the necessary and sufficient *differential* equations which a Ricci tensor must fulfill in order for it to represent the energy tensor of the complex scalar field.

The notation and aims of the present analysis have been explained in previous work,[1] and we will only briefly recapitulate the essential features of the algebraic problem in the next section.

It will be seen that the present problem is analogous to the electromagnetic problem solved by Rainich[2] with an essential difference. The complexion gradient[3] of the complex scalar field is determined by the Ricci tensor only to within a sign.

We present the analysis only for the general field, ignoring the null and degenerate special cases mentioned in the algebraic analysis.[1] The reason for doing so is to avoid unnecessary formal complications. The special cases may be examined separately, as is most profitably done in the electromagnetic problem.[3]

II. REVIEW OF ALGEBRAIC PROBLEM

In the previous paper,[1] we displayed certain algebraic conditions which ensured that the Ricci tensor be of the form

$$R_{\mu\nu} = A_\mu A_\nu + B_\mu B_\nu, \tag{1}$$

where A_μ and B_ν are independent vector fields. We also noted that $R_{\mu\nu}$ was unchanged by the duality rotations,

$$A_\mu \to A'_\mu = A_\mu \cos\theta + B_\mu \sin\theta, \tag{2}$$

$$B_\mu \to B'_\mu = -A_\mu \sin\theta + B_\mu \cos\theta. \tag{3}$$

Indeed, the duality invariance of $R_{\mu\nu}$ shows that we may pick the vector fields in a special way by fixing the value of the complexion angle arbitrarily at some point in space–time. Thus, consider the two scalars

$$I_1 \equiv A^\alpha A_\alpha - B^\alpha B_\alpha, \tag{4}$$

$$I_2 \equiv 2A^\alpha B_\alpha, \tag{5}$$

which under duality rotations transform as follows:

$$I'_1 = I_1 \cos 2\theta + I_2 \sin 2\theta, \tag{6}$$

$$I'_2 = -I_1 \sin 2\theta + I_2 \cos 2\theta. \tag{7}$$

Without affecting the Ricci tensor, then, we may pick the vector fields in some special way by choosing, for example,

$$2A'^\alpha B'_\alpha = I'_2 = 0. \tag{8}$$

The *general* vector fields which are determined by the algebraic restrictions on $R_{\mu\nu}$ may then be obtained from the *extremal* vector fields A'_μ and B'_μ by an arbitrary duality rotation.

The point is that we may assume that A_μ and B_ν are orthogonal. We lose no information by our assumption, since the general vector field solutions of the algebraic problem are obtained by duality rotation. Thus, we drop the prime henceforth, and assume A_μ and B_μ are orthogonal.

We can express our extremal vector fields in terms of the Ricci tensor directly. In so doing, we need only make the (arbitrary) choice of taking A_μ to be spacelike and B_μ to be timelike. That is, we specify

$$A^\mu A_\mu > 0, \qquad B^\mu B_\mu < 0. \tag{9}$$

If we define the tensor $S_{\alpha\beta}$ by

$$S_{\alpha\beta} = 2R_\alpha^\omega R_{\omega\beta} - RR_{\alpha\beta}, \tag{10}$$

then our extremal vector fields are given by the expressions,

$$2A_\alpha A_\beta = R_{\alpha\beta} + S^{-\frac{1}{2}} S_{\alpha\beta}, \tag{11}$$

$$2B_\alpha B_\beta = R_{\alpha\beta} - S^{-\frac{1}{2}} S_{\alpha\beta}. \tag{12}$$

---

[1] R. Penney, J. Math. Phys. 7, 479 (1966).
[2] G. Y. Rainich, Trans. Am. Math. Soc. 27, 106 (1925).
[3] J. A. Wheeler, *Geometrodynamics* (Academic Press Inc., New York, 1962), p. 225.

In the expressions, S is the trace of $S_{\alpha\beta}$, and the positive square root is intended. The important point is that quadratic expressions in either vector field are expressible solely in terms of the Ricci tensor.

It is important to note that the Ricci tensor is also invariant under duality *reflections*. That is, if we change the sign of A_μ or B_μ or both, $R_{\mu\nu}$ is unaffected. It follows that any expression which is *not* invariant under such duality reflections *cannot* be written in terms of the Ricci tensor.

III. DIFFERENTIAL PROPERTIES

If one has solved the algebraic problem for the Ricci tensor, one has deduced that $R_{\mu\nu}$ may be written in terms of the two independent vector fields. If we continue to denote by A_μ and B_ν the extremal vectors which are orthogonal, then the algebraic characterization of $R_{\mu\nu}$ tells us only that

$$R_{\mu\nu} = A'_\mu A'_\nu + B'_\mu B'_\nu, \tag{13}$$

where the general vectors A'_μ, B_ν are obtainable by an arbitrary duality rotation from the extremal fields.

As we have previously remarked,[1] in order for $R_{\mu\nu}$ to represent the complex scalar field, we must have the general vectors obey

$$A'_{\mu\,|\,\nu} - A'_{\nu\,|\,\mu} = 0, \tag{14}$$

$$B'_{\mu\,|\,\nu} - B'_{\nu\,|\,\mu} = 0, \tag{15}$$

$$A'^{|\mu}_{\mu} = 0, \tag{16}$$

$$B'^{|\mu}_{\mu} = 0. \tag{17}$$

By inserting the expressions given in Sec. II for the general fields in terms of the extremal fields, we see that we must have:

$$A_{\alpha\,|\,\beta} - A_{\beta\,|\,\alpha} + B_\alpha\theta_{|\beta} - B_\beta\theta_{|\alpha} = 0, \tag{18}$$

$$B_{\alpha\,|\,\beta} - B_{\beta\,|\,\alpha} - A_\alpha\theta_{|\beta} + A_\beta\theta_{|\alpha} = 0, \tag{19}$$

$$A_\alpha^{\ |\alpha} + B_\alpha\theta^{|\alpha} = 0, \tag{20}$$

$$B_\alpha^{\ |\alpha} - A_\alpha\theta^{|\alpha} = 0. \tag{21}$$

The object is now to interpret the above equations as restrictions on the Ricci tensor. The point is that we want to show that, if we impose certain differential equations, which $R_{\mu\nu}$ must fulfill, we must thereby imply that the extremal fields A_α and B_β and the complexion gradient, $\theta_{|\alpha}$, obey the above set of differential equations.

At this point, then, we want to find a set of equations which contain only the Ricci tensor, but which are equivalent to the restrictions on the extremal fields and complexion gradient. In so doing, we are examining the general case; we will assume that

$$(A^\alpha A_\alpha)^2(B^\lambda B_\lambda)^2 > 0. \tag{22}$$

It happens that the analysis is quite tedious, and it is expedient to adopt many useful notations. We define the following:

$$A_{\alpha\,|\,\beta} - A_{\beta\,|\,\alpha} \equiv A_{\alpha\beta}, \qquad A_\alpha^{\ |\alpha} \equiv a, \tag{23}$$

$$B_{\alpha\,|\,\beta} - B_{\beta\,|\,\alpha} \equiv B_{\alpha\beta}, \qquad B_\alpha^{\ |\alpha} \equiv b, \tag{24}$$

$$B^\alpha A_{\alpha\beta} \equiv a_\beta, \qquad \theta_{|\alpha} \equiv \theta_\alpha, \tag{25}$$

$$A^\alpha B_{\alpha\beta} \equiv b_\beta, \qquad A^\alpha A_\alpha \equiv A^2, \qquad B^\alpha B_\alpha = B^2. \tag{26}$$

Our conditions involving θ_α are then as follows, with the new notation,

$$A_{\alpha\beta} + B_\alpha\theta_\beta - B_\beta\theta_\alpha = 0, \tag{18}$$

$$B_{\alpha\beta} - A_\alpha\theta_\beta + A_\beta\theta_\alpha = 0, \tag{19}$$

$$a + B_\alpha\theta^\alpha = 0, \tag{20}$$

$$b - A_\alpha\theta^\alpha = 0. \tag{21}$$

With little effort, using the orthogonality of the extremal fields, we may rewrite our conditions, separating the information concerning θ_α and the restrictions on the extremal fields. We obtain the equivalent set:

$$2A^2B^2\theta_\beta = B^2[b_\beta + bA_\beta] - A^2[a_\beta + aB_\beta], \tag{27}$$

$$0 = B^2[b_\beta + bA_\beta] + A^2[a_\beta + aB_\beta], \tag{28}$$

$$A^2B_{\alpha\beta} = A_\alpha b_\beta - A_\beta b_\alpha, \tag{29}$$

$$B^2A_{\alpha\beta} = B_\alpha a_\beta - B_\beta a_\alpha, \tag{30}$$

by simple algebra. The latter set of equations are not all independent, however, as a count of equations shows.

To proceed in a reasonable manner, we need to realize that quadratic combinations of either vector field may be expressed in terms of the Ricci tensor. Thus we find it useful to define the following tensors which are expressible solely in terms of the Ricci tensor:

$$R_{A\alpha\beta} \equiv A_\alpha A_\beta, \quad G_{A\alpha\beta} \equiv A_\alpha A_\beta - \tfrac{1}{2}A^2 g_{\alpha\beta}, \tag{31}$$

$$R_{B\alpha\beta} \equiv B_\alpha B_\beta, \quad G_{B\alpha\beta} \equiv B_\alpha B_\beta - \tfrac{1}{2}B^2 g_{\alpha\beta}, \tag{32}$$

where A and B are labels, and not tensor indices.

IV. RESTRICTIONS ON EXTREMAL FIELDS

In this section, we want to write the differential conditions which do not involve θ_α as conditions on the Ricci tensor. First of all, we consider Eqs. (29)

and (30). We use the following *identities*:

$$R^{\alpha}_{A\rho}(G_{A\alpha\beta\,|\,\gamma} - G_{A\alpha\gamma\,|\,\beta}) \equiv A^2 A_\rho A_{\beta\gamma}, \qquad (33a)$$

$$R^{\alpha}_{B\rho}(G_{B\alpha\beta\,|\,\gamma} - G_{B\alpha\gamma\,|\,\beta}) \equiv B^2 B_\rho B_{\beta\gamma}, \qquad (33b)$$

which are true solely because of the vector composition of the ingredient tensors.

We also use the *identities:*

$$R^{\omega\tau}_{B} R^{\lambda\sigma}(G_{B\omega\sigma\,|\,\nu} - G_{B\omega\nu\,|\,\sigma}) \equiv B^2 A^\lambda B^\tau b_\nu, \qquad (34a)$$

$$R^{\omega\tau}_{A} R^{\lambda\sigma}_{B}(G_{A\omega\sigma\,|\,\nu} - G_{A\omega\nu\,|\,\sigma}) \equiv A^2 B^\lambda A^\tau a_\nu, \qquad (34b)$$

which, again, are true without assuming any properties for the vectors involved.

Using the mentioned identities, we may immediately write Eqs. (29) and (30) in terms of the Ricci tensor. For brevity, we examine only Eq. (29). Multiplying Eq. (29) by B_ρ we have

$$A^2 B_\rho B_{\alpha\beta} = B_\rho(A_\alpha b_\beta - A_\beta b_\alpha) \qquad (35)$$

from which the original equation is retrievable by contraction with B^ρ. We then use Eqs. (33b) and (34a) and obtain directly the condition

$$^{R}A^{R^\alpha}B\rho(G_{B\alpha\beta\,|\,\gamma} - G_{B\alpha\gamma\,|\,\beta})$$
$$= R^{\omega}_{B\rho}[R^{\sigma}_{A\beta}(G_{B\omega\sigma\,|\,\gamma} - G_{B\omega\gamma\,|\,\sigma}) - R^{\sigma}_{A\gamma}(G_{B\omega\sigma\,|\,\beta} - G_{B\omega\beta\,|\,\sigma})]. \qquad (36)$$

Similarly, the transcription of Eq. (30) results in an equation exactly analogous to Eq. (36) with the labels A and B interchanged. We thus have accomplished our aim of writing Eqs. (29) and (30) in terms of the Ricci tensor, without losing any information.

To complete the aim of the present section, we want to write Eq. (28) solely in terms of the Ricci tensor. Unfortunately, to do so, we must follow a circuitous route. Thus, it is worthwhile to outline the method of attack. We will return to the Eqs. (28)–(30), and realize that (29) and (30) have been paraphrased in terms of the Ricci tensor. We then show by combining the information in our set of equations that the Bianchi identities are therein contained. Conversely, since the Bianchi identities are trivially satisfied by *any* Ricci tensor, we deduce certain identities which must be fulfilled by the external fields. We then use the latter identities to write Eq. (28) in a form suitable for treatment.

If we examine Eq. (28), and contract first with A^β and then with B^β, we obtain the deductions

$$B^2 b = -A^\beta a_\beta, \qquad (37)$$

$$A^2 a = -B^\beta b_\beta, \qquad (38)$$

which we proceed to use in Eqs. (29) and (30). We multiply Eq. (29) by $B^2 B^\alpha$ and Eq. (30) by $A^2 A^\alpha$ and add the resulting equations, obtaining the relation

$$A^2 B^2 (B^\alpha B_{\alpha\beta} + A^\alpha A_{\alpha\beta}) = A^2 B^2 (a A_\beta + b B_\beta), \qquad (39)$$

which we simplify because $A^2 B^2$ does not vanish.

Equation (39) is then easily written in the form

$$(R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta})^{|\alpha} = 0 \qquad (40)$$

which are the Bianchi identities. Conversely, since the Ricci tensor identically obeys the Bianchi identities, and since Eqs. (29) and (30) are deductions from conditions on the Ricci tensor, we may *deduce* Eqs. (37) and (38) from properties of the Ricci tensor. Indeed, the Bianchi identities *alone* imply the scalar conditions of Eqs. (37) and (38).

The upshot is that we have lost no information in writing Eq. (28) in the following form:

$$0 = \{B^2 b_\beta - A_\beta A^\rho a_\rho\} + \{A^2 a_\beta - B_\beta B^\rho b_\rho\}, \qquad (41)$$

which is well suited to analysis. Proceeding, we directly obtain the form

$$0 = (B^2 g^{\omega}_{\beta} - B_\beta B^\omega) b_\omega + (A^2 g^{\omega}_{\beta} - A_\beta A^\omega) a_\omega \qquad (42)$$

by elementary manipulation of dummy indices.

To aid our analysis, we use the following relations, which assume no special properties of the extremal fields except orthogonality:

$$A^\lambda B^\delta [G_{A\lambda\delta\,|\,\omega} - G_{A\lambda\omega\,|\,\delta}] = A^2 a_\omega, \qquad (43a)$$

$$B^\lambda A^\delta [G_{B\lambda\delta\,|\,\omega} - G_{B\lambda\omega\,|\,\delta}] = B^2 b_\omega. \qquad (43b)$$

If we then define the combinations

$$F_{A\alpha\beta} \equiv A_\alpha A_\beta - A^2 g_{\alpha\beta}, \qquad (44a)$$

$$F_{B\alpha\beta} \equiv B_\alpha B_\beta - B^2 g_{\alpha\beta}, \qquad (44b)$$

and multiply Eq. (42) by $A^2 B^2 B^\tau A^\sigma$, we read as follows:

$$0 = A^2 F^{\omega}_{B\beta} R^{\tau\lambda}_{B} R^{\sigma\delta}_{A} (G_{B\lambda\delta\,|\,\omega} - G_{B\lambda\omega\,|\,\delta})$$
$$+ B^2 F^{\omega}_{A\beta} R^{\delta\tau}_{B} R^{\sigma\lambda}_{A} (G_{A\lambda\delta\,|\,\omega} - G_{A\lambda\omega\,|\,\delta}). \qquad (45)$$

This latter equation is expressed solely in terms of the Ricci tensor since it contains only quadratic combinations of either vector field, and the extremal vector fields are expressed solely in terms of the Ricci tensor, as we saw previously. Of course, we could write Eq. (45) explicitly in terms of $R_{\mu\nu}$ by using the expressions for the extremal fields, but it is not necessary to do so, and would lead to extremely complicated expressions.

We have accomplished our purpose of writing the restrictions on the extremal fields solely in terms of the Ricci tensor. If we had a Ricci tensor which obeyed the algebraic conditions previously displayed,[1] we would find that it would have to obey the restrictions given by Eqs. (36) and (45) [and the equation obtained from (36) by interchanged labels A and B] in order to represent the complex scalar field. Conversely, if the Ricci tensor obeyed the mentioned restrictions, then we would be able to deduce that the extremal fields, of which $R_{\mu\nu}$ was composed, obeyed the Eqs. (28)–(30).

V. THE COMPLEXION GRADIENT

We wish to consider the expression for the complexion gradient, which is

$$2A^2B^2\theta_\beta = B^2[b_\beta + bA_\beta] - A^2[a_\beta + aB_\beta]. \quad (27)$$

Before doing any manipulations, it is important to note that θ_β cannot be expressed solely in terms of the Ricci tensor. Indeed, if we change the sign of one of the extremal fields by a duality reflection, then the complexion gradient changes sign according to the above relation. Therefore, θ_β cannot be formed from the Ricci tensor and its derivatives, since the Ricci tensor is invariant under duality reflections.

In this respect, the complexion gradient is different in character from that which arises in the electromagnetism case.[2] The essential difference in character between the complex scalar field and the electromagnetic field is that the extremal fields A_α and B_β are truly independent vector fields, whereas the Maxwell field tensor and its dual are linearly related.[3]

We wish to determine to what extent the complexion gradient *is* determined by the Ricci tensor. To do so, we use again the relations,

$$B^2b = -A^\beta a_\beta, \quad (37)$$

$$A^2a = -B^\beta b_\beta, \quad (38)$$

and elementary manipulations to obtain

$$2A^2B^2\theta_\beta = P^{\omega}_{B\beta}b_\omega - P^{\omega}_{A\beta}a_\omega. \quad (46)$$

Here we have defined the tensors;

$$P_{A\omega\beta} = A_\omega A_\beta + A^2 g_{\omega\beta}, \quad (47a)$$

$$P_{B\omega\beta} = B_\omega B_\beta + B^2 g_{\omega\beta}, \quad (47b)$$

which are expressible solely in terms of the Ricci tensor.

We again use the relations given by Eq. (43a) and (43b), multiply by A^2B^2 and obtain,

$$2(A^2B^2)^2\theta_\beta = A^\lambda B^\delta Q_{\lambda\delta\beta}, \quad (48)$$

where the tensor $Q_{\lambda\delta\beta}$ is expressible solely in terms of the Ricci tensor and is explicitly,

$$Q_{\lambda\delta\beta} \equiv R_A P^{\omega}_{B\beta}(G_{B\delta\lambda\,|\,\omega} - G_{B\delta\omega\,|\,\lambda}) - R_B P^{\omega}_{A\beta}(G_{A\lambda\delta\,|\,\omega} - G_{A\lambda\omega\,|\,\delta}). \quad (49)$$

To complete our analysis, we form the tensor product of θ_β with θ_α and have, finally, the expression

$$\theta_\beta\theta_\alpha = H_{\beta\alpha}, \quad (50)$$

where the very complicated symmetric tensor $H_{\alpha\beta}$ is defined by the expression,

$$[2(A^2B^2)^2]^2 H_{\beta\alpha} \equiv R^{\lambda\rho}_A R^{\delta\tau}_B Q_{\lambda\delta\beta} Q_{\rho\tau\alpha}, \quad (51)$$

which could be simplified somewhat by using the algebraic properties of the Ricci tensor.

The important point is, however, that the complexion gradient θ_α is only implicitly expressible in terms of the Ricci tensor. $H_{\alpha\beta}$ is expressible directly in terms of the Ricci tensor and allows one to determine θ_α to within a sign.

Our analysis is not complete unless we specify conditions on the tensor $H_{\alpha\beta}$ which ensure that it is always expressible in terms of a vector product of scalar gradients which play the role of the complexion gradient.

However, fortunately, it has been shown[4] that the necessary and sufficient conditions which a symmetric tensor must obey in order to be equivalent to a vector product of scalar gradients are as follows:

$$H^{\mu\beta}H_{\alpha\beta} = HH^{\mu}_{\alpha}, \quad (52a)$$

$$H_{00} > 0, \quad (52b)$$

$$H < 0, \quad (52c)$$

$$H^{\alpha}_{\beta}[(H_{\alpha\beta} - \tfrac{1}{2}Hg_{\alpha\beta})_{|\gamma} - (H_{\alpha\gamma} - \tfrac{1}{2}Hg_{\alpha\gamma})_{|\beta}] = 0, \quad (52d)$$

and therefore, these are the conditions which are necessary and sufficient to allow us to determine the complexion gradient at any point to within a sign.

VI. AMBIGUITY OF THE COMPLEXION

The fact that the complexion gradient is always ambiguous to within a sign at any point where the Ricci tensor is known is *not* a feature which spoils the Rainich geometrization scheme. Indeed, the ambiguity is of the same character as that which occurs in the Rainich treatment of the electromagnetic field.

If we define the complexion[3] of the complex scalar field as is done for the Maxwell field, viz.,

$$\theta \equiv \int_0^x \theta_\beta\, dx^\beta + \theta_0, \quad (53)$$

where θ_0 is some arbitrary value at a selected point, and we integrate to any arbitrary point x^α, then the ambiguity of sign for the complexion gradient results in an ambiguity of the complexion.

[4] R. Penney, J. Math. Phys. **6**, 1029 (1965).

However, we must recall that the extremal fields A_ρ and B_λ are themselves ambiguous to within a sign, just as for the Maxwell field. Similarly, the value of θ_0 is ambiguous to within a sign.

We have so far shown that, if a Ricci tensor is to represent a complex scalar field, it is *necessary* that the equations of the preceding sections be fulfilled.

Conversely, if the mentioned algebraic and differential equations are obeyed by the Ricci tensor, those conditions are *sufficient* to determine the complex scalar field. To see this, realize that the algebraic conditions given in a previous paper[1] allow us to determine the vector fields to within a duality rotation and a sign. We obtain from the *algebra* the local values of the vector fields,

$$A'_\mu = \epsilon_A A_\mu \cos\theta + \epsilon_B B_\mu \sin\theta, \tag{54a}$$

$$B'_\mu = -\epsilon_A A_\mu \sin\theta + \epsilon_B B_\mu \cos\theta, \tag{54b}$$

where ϵ_A, ϵ_B are the sign ambiguities of the extremal fields, and θ is an arbitrary angle.

The differential conditions, which were expressed in terms of $H_{\alpha\beta}$ above, allow one to determine the complexion gradient to within a sign. As a consequence, we obtain the complexion angle throughout space–time to be

$$\theta = \epsilon_\theta\, |\theta| + \theta_0, \tag{55}$$

where we denote by ϵ_θ the sign ambiguity of the integral of the complexion gradient. At this point, however, we must realize that the ambiguity of the complexion angle is reduced by the notion of continuity. If θ has the value θ_0 at some point, then its nearby values must be only infinitesimally different. Thus, only a global ambiguity of the form ϵ_θ remains. That is, starting from some standard point at which the complexion angle is θ_0, any particular observer will choose a sign for θ_α, and will maintain that choice on moving from the standard point. Therefore, depending upon the original choice of sign for θ_α, only two global situations will arise, which is implied by Eq. (50).

The important point is that we can instruct any geometer as to which sign to adopt for the various choices mentioned. Whether or not our instructions are physically realistic is a separate problem which can only be answered by interpretations which lie outside the realm of classical geometrodynamics.

Indeed, if we realize that, within the context of quantum field theory, the vector fields A_μ and B_α represent the gradients of the field amplitudes for the π^+ or π^- mesons; we will appreciate that the sign of either extremal field is irrelevant. Likewise, the possibility of two global situations for the complexion angle as a function of position may be viewed as an expression of the fact that a given geometer may make a choice of which extremal field is to represent π^+ and which is to represent π^-. Of course, in the absence of interactions, such a choice is a matter of definition.

The conclusion must be that the ambiguities of signs we have noted can only be resolved by conventions and definitions. To appraise whether such conventions are consistent, one must consider the problem of geometrizing interacting fields.

For our present purposes, we can resolve the ambiguities arbitrarily by instructing a geometer to adopt the positive sign of the square-root expressions which arise in calculating the extremal fields or the complexion gradient. That is, in calculating either of the extremal fields, or the complexion gradient, the geometer necessarily must perform the operation of taking the square root of a component of a tensor formed from the Ricci tensor. For example, in calculating θ_α, the geometer needs to decide whether to use

$$\theta_0 = +(2H_{00})^{\frac{1}{2}} \tag{56}$$

or the opposite sign. If we adopt the convention of instructing the geometer always to use the positive sign where such choices are to be made, then no ambiguity remains.

VII. CONCLUSIONS

We have shown that the complex scalar field, which may loosely be considered as representing massless charged pions, can be completely geometrized in the general case. We have *not* considered the various null or degenerate cases which may arise. We have *not* shown that our conventions for resolving the ambiguities of sign which arise in the problem are physically reasonable or consistent with the interpretation of the complex scalar field.

The necessary and sufficient conditions that a Ricci tensor must fulfill to represent the energy tensor of a complex scalar field are given by Eqs. (36) (and its relabeled form), (45), and (52), together with the algebraic conditions previously derived.[1] In addition, one must adopt the convention that positive square roots are implied wherever a choice is to be made in the analysis.

It may be that the null and degenerate special cases of the complex scalar field may be similarly treated, but that remains to be shown.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Tetrad Formulation of Junction Conditions in General Relativity*

FRANK B. ESTABROOK AND HUGO D. WAHLQUIST
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California

(Received 23 August 1966)

Explicit and complete junction conditions across internal boundaries are found for the Ricci rotation coefficients and Riemann tensor components in an arbitrary reference tetrad frame; the reference frame may be discontinuous at the boundary. The allowed jumps in inertial and gravitational field dyadics are given for the case of normal boundary motion. When the reference frame is chosen to be comoving with the matter distribution on either side of the boundary, the junction conditions include such immediately physical results as the relativistic Rankine–Hugoniot relations. It is expected that use of such comoving reference frames will be important in finding discontinuous exact solutions.

I. INTRODUCTION

ADMISSIBLE coordinates χ^μ, in the Class C^2 differentiable manifold of space–time, have been carefully defined by Lichnerowicz.[1] In such coordinate systems, he postulates (a) that the metric tensor $g_{\mu\nu}(\chi^\mu)$ is continuous of Class C^1, and further (b) that the first partials $g_{\mu\nu,\sigma}$ (or, equivalently, the $\Gamma^\mu_{\nu\sigma}$) are *piecewise* continuous of Class C^2. Postulate (a) evidently ensures the existence of interval invariants and of parallel transport in every neighborhood. In the terminology of Synge,[2] it ensures elementary flatness. It ensures the existence of a second fundamental form for any imbedded 3-surface Σ, and ensures the possibility of construction of Gaussian coordinates based on Σ. O'Brien and Synge[3] have shown how postulate (b) leads (at internal "boundaries" such as Σ) to physically interpretable junction requirements on components of the Riemann tensor. In particular, for those components identified by the field equations as describing energy and momentum density and flux, the junction conditions are the relativistic Rankine–Hugoniot relations.[2,4] They ensure that no sources of energy and momentum exist as singular distributions on internal boundaries. Similar local physical interpretations, as junction conditions, may in fact be made of all the implications of postulate (b).

The introduction and actual use of admissible coordinates in formulating general relativistic models is usually inconvenient, especially at boundaries across which the postulated physical configuration changes. It is in practice much more appropriate to use different, intrinsically defined, anholonomic reference frames on either side of such boundaries. What is then needed is a complete formulation of all the junction conditions required to be imposed, conditions which together are of course equivalent to the Lichnerowicz postulates. The lack of such an explicit formulation may be a contributory reason for the paucity of exact general relativistic solutions having physical internal discontinuities.

The Lichnerowicz postulate (a) of continuity of metric structure clearly allows us to have present in the space–time, a Class C^0 tetrad frame: a set of four continuous vector fields ${}_R\Lambda^\mu$ (labeled by $R = 0, 1, 2, 3$) such that everywhere ${}_R\Lambda^\mu g_{\mu\nu}\, {}_S\Lambda^\nu = \eta_{RS}$, or ${}_R\Lambda^\mu \eta^{RS}\, {}_S\Lambda^\nu = g^{\mu\nu}$, *where the* η_{RS} *and the reciprocal* η^{RS} *are constants* (and so indeed continuous of Class C^∞!). In a tetrad formulation of general relativity, the $g_{\mu\nu}$, components of the metric tensor in the holonomic admissible coordinate frame, are replaced by the η_{RS}, anholonomic components. The ${}_R\Lambda^\mu$ make explicit the local metric, its signature, and the local meaning of parallelness. In the following, we usually specialize to an orthonormal frame, in which $\eta_{RS} = \eta^{RS} = \text{diag}\ (-1, 1, 1, 1)$. Postulate (b) allows the fields ${}_R\Lambda^\mu$ to be taken also piecewise continuous and covariantly differentiable of Class C^3. Across an internal boundary, a 3-surface Σ, we may then only allow jumps in the first, and higher, derivatives of the ${}_R\Lambda^\mu$ *normal* to Σ.

We will call the continuous ${}_R\Lambda^\mu$ the *standard* orthonormal tetrad frame. We go on in Sec. II, to consider also a piecewise C^3 continuous *reference* tetrad frame, the vectors ${}_r\lambda^\mu$ of which again satisfy the orthonormality conditions, but are *not* C^0, being related across Σ by arbitrary orthogonal transformation. The desired physical junction conditions then result from the expression, in this second, discontinuous, tetrad frame, of the tangential continuity requirements for the standard fields ${}_R\Lambda^\mu$.

In Sec. III we consider in detail the specialized case when the surface of discontinuity separates two reference frames whose relative motion is normal to it; this case promises to be of considerable practical use

* This work was sponsored by the National Aeronautics and Space Administration under Contract No. NAS7-100.
[1] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de L'Électromagnétisme* (Masson & Cie, Paris, 1955).
[2] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960).
[3] S. O'Brien and J. L. Synge, "Jump Conditions at Discontinuities In General Relativity," Commun. Dublin Inst. Adv. Stud., A, 9 (1953).
[4] A. H. Taub, Phys. Rev. 74, 328 (1948).

in space–times with symmetries. We now can conveniently put the junction conditions into dyadic notation[5,6]; there are 29 relations across Σ between the physical 3-vectors and dyadics of the inertial-reference fields $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$, S (acceleration, triad, rotation, vorticity, and rate-of-strain), the stress, momentum density and energy density fields T, $\mathbf{t}$, ρ, the gravitational fields A and B, and the orthonormal 3-vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ and shock speed v describing Σ itself. The junction conditions found for $\mathbf{a}$ and $\boldsymbol{\Omega}$ are closely analogous to those for $\mathbf{E}$ and $\mathbf{B}$ in electromagnetism. Those for T, $\mathbf{t}$, and ρ are the relativistic Rankine–Hugoniot relations for general media; for comoving dyadic frames in perfect fluids, they reduce to the known relations between densities and pressures on either side of a shock.[4] The junction conditions for the traceless dyadics A and B are again analogous to the electromagnetic case, although now the source densities of the field are also involved.

In Sec. IV we consider the exceptional case which occurs when Σ is null. *Inter alia*, we of course find in empty space that only type N jumps in A and B are allowed. Lastly, we give in Sec. V a discussion of further relations resulting from postulate (a), which must be satisfied when three or more surfaces Σ join in a 2-manifold.

The results of this paper will in the future be applied to some simple physical cases with spherical symmetry.[7] When the equations of Sec. III are specialized to this case, only seven dyadic junction conditions remain, and the use of intrinsic comoving reference frames in each of various physical regions of a single nonstatic space–time is analytically reasonable.

II. TETRAD JUNCTION CONDITIONS

The standard tetrad fields ${}_R\Lambda^\mu$ are continuous across Σ and are continuous and thrice differentiable parallel to Σ. The first implication of this is that the first tangential derivatives ${}_R\Lambda^\mu{}_{;\sigma}P^\sigma_\nu$ are continuous across Σ, P^σ_ν being the projection operator into Σ. Since we do not wish explicitly to use admissible holonomic coordinates, we "strangle" these expressions with the continuous standard tetrad fields to obtain the scalar conditions

$$ {}_R\Lambda_{\mu;\sigma}P^\sigma_\nu\, {}_S\Lambda^\mu\, {}_T\Lambda^\nu \mathbb{C}. \tag{1}$$

By $\mathbb{C}$ we mean continuous across Σ. Taking account of the projection operators, and the orthonormality of the standard tetrad, it may be seen that there are 18 relations in Eq. (1).

Similarly the second and third tangential derivatives of ${}_R\Lambda^\mu$ are continuous: $[{}_R\Lambda_{\mu;\sigma}P^\sigma_\nu]_{;\tau}P^\tau_\rho$, etc. We will not write most of these, as they are guaranteed as soon as one has obtained sufficiently differentiable functions conforming to Eq. (1). However, *some* combinations of the second partials of a tetrad frame are dignified by new symbols—viz., those which covariantly describe the underlying geometry and do not refer to the inertial structure of the specific frame. These are the Riemann tensor components, formed from just the antisymmetrized second derivatives of any tetrad frame. They have, in Einstein theory, unique physical (dynamical) significance. In actual problems with internal boundaries, it is the behavior of the Riemann components which may well be specified *ab initio*, and the first partials (Ricci rotation coefficients) and metric are found by integration. In short, in addition to Eq. (1) we are only interested in the junction conditions for the scalars formed from the antisymmetrized second derivatives

$$\{{}_R\Lambda_{\mu;[\sigma\tau]}P^\sigma_\nu P^\tau_\rho + {}_R\Lambda_{\mu;\sigma}P^\sigma_{[\nu;|\tau|}P^\tau_{\rho]}\}{}_S\Lambda^\mu\, {}_T\Lambda^\nu\, {}_P\Lambda^\rho \mathbb{C}. \tag{2}$$

There are 14 scalar relations in Eq. (2), which we next find in more elegant form.

Without any loss of generality in describing the continuity properties of our space–time, we may take the standard tetrad frame ${}_R\Lambda^\mu$ to have *one* of its vectors, viz., ${}_2\Lambda^\mu$, aligned normal to the internal boundary Σ. If ${}_2\Lambda^\mu$ is spacelike, Σ is a timelike 3-surface which appears in any local spatial reference frame as a moving 2-surface of discontinuity, or shock front. If ${}_2\Lambda^\mu$ is timelike, Σ is spacelike, and the 2-surface of discontinuity appears to be moving at a speed greater than that of light. (The case when Σ is a null 3-surface will be considered in Sec. IV.) The projection tensor is then

$$P^\sigma_\nu = g^\sigma_\nu - {}_2\Lambda^\sigma\, {}^2\Lambda_\nu. \tag{3}$$

The Bianchi differential identities (which will involve third partials of any set of tetrad vectors) are expressed in conservation form by use of the double-dual Riemann tensor: ${}^*R^{\mu\nu\sigma\tau}{}_{;\tau} = 0$. By considering a Gauss theorem on a limiting pill box including Σ, knowing that the ${}^*R^{\mu\nu\sigma\tau}$ and ${}_S\Lambda_{\mu,\nu}$ can only have finite jumps across Σ, we immediately find the scalar requirements

$$ {}_R\Lambda^\mu\, {}_S\Lambda^\nu\, {}_T\Lambda^\sigma\, {}^*R_{\mu\nu\sigma\tau}\, {}_2\Lambda^\tau \mathbb{C}. \tag{4}$$

Considering the symmetries of the Riemann tensor, there are fourteen conditions in Eq. (4). Introducing the Ricci commutation relation into Eq. (2) and

[5] F. B. Estabrook and H. D. Wahlquist, J. Math. Phys. 5, 1629 (1964).
[6] H. D. Wahlquist and F. B. Estabrook, J. Math. Phys. 7, 894 (1966).
[7] H. D. Wahlquist and F. B. Estabrook, Phys. Rev. 156, 1359 (1967).

taking account of the continuity of the quantities in Eq. (1), one shows that they are precisely the same conditions.

As explained in the introduction, we now introduce a *reference* orthonormal tetrad frame ${}_r\lambda^\mu$, discontinuous across Σ by an orthogonal rotation. The junction conditions across Σ are written in this physical reference frame; Eqs. (1) and (4) become, respectively,

$$({}_R\Lambda_{u,s} + \Gamma^{\cdot q\cdot}_{s.u}\,{}_R\Lambda_q)({}_T\Lambda^s - {}_2\Lambda^s\eta^{\cdot 2}_{T})_S\Lambda^u\mathbb{C} \tag{5}$$

and

$${}_R\Lambda^r{}_S\Lambda^s{}_T\Lambda^t\,{}^*R_{prst}\,{}_2\Lambda^p\mathbb{C}. \tag{6}$$

The $\Gamma^{\cdot q}_{s.u}$ are the Ricci rotation coefficients of the reference tetrad frame, in the notation of Ref. 5.

Regarded as junction conditions on the physical variables Γ_{rst} and ${}^*R_{prst}$, the 32 relations (5) and (6) are still in part implicit, in that they involve the reference tetrad components of the "standard" tetrad fields, ${}_R\Lambda^r$. These now need only to be known on, and differentiated in, Σ (but given twice—in the reference tetrad frame as we approach from either side). The ${}_R\Lambda^r$ should now be regarded as just orthogonal transformation matrices taking the ${}_r\lambda^\mu$ tetrad on either side into a convenient "standard" orientation ${}_R\Lambda^\mu = {}_R\Lambda^r{}_r\lambda^\mu$ implicitly defined on Σ. The tetrad components of a vector V^μ change according to $V^R = {}^R\Lambda_r V^r$. The 4×4 matrix $\mathbf{\Lambda} = {}_R\Lambda^r$ satisfies $\mathbf{\Lambda}\boldsymbol{\eta}\mathbf{\Lambda}^T = \boldsymbol{\eta}$, and so would have in general 6 parameters on each side of Σ. But we have taken ${}_2\Lambda^r$ to be the unit normal to Σ. This means that, regarding Σ as given, on either side we still have in the transformation matrix 3 *a priori* unknown parameters, functions of position on Σ; also that ${}_2\Lambda^\mu$ itself, being 3-normal, must satisfy ${}_2\Lambda_{[\mu}\,{}_2\Lambda_{\nu;\sigma]} = 0$:

$$\epsilon^{prst}\,{}_2\Lambda_r[{}_2\Lambda_{s,t} + \Gamma^{\cdot q}_{t.s}\,{}_2\Lambda_q] = 0. \tag{7}$$

III. NORMAL BOUNDARY MOTION

In most practical problems the 32 junction relations, Eqs. (5) and (6), are simplified by symmetries. In particular, we wish now to consider how these relations reduce in what we might call the case of normal boundary motion. Seen in space–time, this is when the timelike reference vectors ${}_0\lambda^\mu$ on either side of Σ are coplanar with ${}_2\Lambda^\mu$, the normal to Σ. The "reference fluid" ${}_0\lambda^\mu$ on either side of Σ is seen, in the local spatial reference frame on the other side of Σ, to be moving with 3-velocity normal to the instantaneous bounding 2-surface, or shock front. It is clear, for example, that this will be the situation in problems with plane, spherical, or cylindrical symmetry, if one naturally adopts, on both sides of a plane, spherical or cylindrical symmetrical boundary, timelike reference vector congruences which themselves reflect the same symmetry.

So we now choose for further discussion the following particular form for the orthogonal transformation matrices ${}_R\Lambda^r$:

$${}_R\Lambda^r = \begin{array}{cc} & R = \\ \begin{pmatrix} \gamma & \gamma v\hat{\mathbf{v}} \\ 0 & \hat{\mathbf{u}} \\ \gamma v & \gamma\hat{\mathbf{v}} \\ 0 & \hat{\mathbf{w}} \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array} \tag{8}$$

$$r = 0,\ 1,\ 2,\ 3,$$

where $\gamma = (1 - v^2)^{-\frac{1}{2}}$, and $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ are an orthonormal triad. We use a 3-vector symbolism for spacelike indices $r = 1, 2, 3$—this is the usual dyadic convention. We note that ${}^R\Lambda_r$ applied to the unit timelike vector $l^r = (1, 0, 0, 0)$ gives standard components $l^R = (\gamma, 0, -\gamma v, 0)$ while a unit spacelike vector $(0, \hat{\mathbf{v}})$ becomes $(-\gamma v,\ 0,\ \gamma,\ 0)$. Further $(0,\ \hat{\mathbf{u}})$ becomes $(0, 1, 0, 0)$; $(0, \hat{\mathbf{w}})$ becomes $(0, 0, 0, 1)$. So Eq. (8) is explicitly a 4-screw[2] with one of its canonical 2-flats in Σ. As seen in the reference tetrad frame, the transition to the "standard" frame is a Lorentz transformation in the $\hat{\mathbf{v}}$ direction, combined with a pure rotation about that direction. *The $\hat{\mathbf{v}}$ direction is the instantaneous normal to the shock front, and v is the normal speed of that front.* The 3-vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ are in the instantaneous front.

The transformation from the standard frame to the reference frame on the other side of Σ, will be a 4-screw *with the same canonical 2-flats*. The shock speed v is, of course, different on either side of the front. The 3-vector triads $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ are to be physically identified across the front—but this is not to say that the *components* of these unit vectors with respect to the spatial reference vectors are unchanged in crossing the front.

Substituting Eq. (8) into Eq. (7), we find on either side of the boundary

$$(2v^2 - 1)(\mathbf{\Omega} - \hat{\mathbf{v}}\cdot\mathbf{\Omega}\hat{\mathbf{v}}) + \hat{\mathbf{v}} \times [\gamma^{-1}\hat{\mathbf{v}}' + \nabla v + \mathbf{S}\cdot\hat{\mathbf{v}} + v\mathbf{a}] = 0 \tag{9}$$

and

$$\hat{\mathbf{v}}\cdot[\nabla \times \hat{\mathbf{v}} + 2v\mathbf{\Omega}] = 0. \tag{10}$$

Derivatives comoving in the shock front are denoted by a prime: $\phi' = \gamma(\dot{\phi} + v\hat{\mathbf{v}}\cdot\nabla\phi)$, $\hat{\mathbf{u}}' = \gamma(\dot{\hat{\mathbf{u}}} + \boldsymbol{\omega} \times \hat{\mathbf{u}} + v\hat{\mathbf{v}}\cdot\nabla\hat{\mathbf{u}})$, etc. The prime operation is the covariant absolute differentiation operation in Σ, in the ${}_0\Lambda^\mu$ direction. It has the same dyadic form on either side of Σ, a form which is to be physically identified across Σ.

Most of the scalar continuity conditions can now be conveniently written as conditions of continuity of tangential components ($\mathbb{C}_T$) or of normal components ($\mathbb{C}_N$) of 3-vectors [and symmetric dyadics]. In the case $\mathbb{C}_T$ we will understand that when the vectors [dyadics] are operated on by $\hat{\mathbf{u}}\cdot$ or $\hat{\mathbf{w}}\cdot$ [by $\hat{\mathbf{u}}\hat{\mathbf{u}}:$, $\hat{\mathbf{w}}\hat{\mathbf{w}}:$ or $(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}}):$] the resulting scalars are $\mathbb{C}$. In the case $\mathbb{C}_N$ that when the vectors [dyadics] are operated on by $\hat{\mathbf{v}}\cdot$ [by $\hat{\mathbf{v}}\hat{\mathbf{v}}:$] the resulting scalars are $\mathbb{C}$. After each continuity symbol, $\mathbb{C}_T$ or $\mathbb{C}_N$, we will also indicate the number of such scalar conditions implied.

Using Eqs. (9) and (10), we find 15 conditions remaining in Eq. (5): first

$$\hat{\mathbf{v}}\cdot\nabla\times(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}})\mathbb{C}_T(2), \tag{11}$$

$$(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}})'\mathbb{C}_T(1). \tag{12}$$

These are in fact the only continuity relations of our set which explicitly involve the reference vector orientation in the front; this orientation is described by the dyadic $\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}}$ and (like the vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ themselves) may be physically identified as continuous across the front. For simple situations Eqs. (11) and (12) will be trivially satisfied.

$$\gamma(\nabla\hat{\mathbf{v}} + \hat{\mathbf{v}}\nabla) + 2\gamma v\mathsf{S}\mathbb{C}_T(3). \tag{13}$$

These are continuity requirements for the instantaneous second fundamental form of the 2-front, as seen from either side.

$$\gamma v(\nabla\hat{\mathbf{v}} + \hat{\mathbf{v}}\nabla) + 2\gamma\mathsf{S}\mathbb{C}_T(3). \tag{14}$$

While again involving the second fundamental form, this is best regarded as a continuity requirement on S, the rate-of-strain dyadic. S, together with $\mathbf{a}$ and $\mathbf{\Omega}$, describes the inertial field in the explicitly physical dyadic formalism. These latter vectors have tangential and normal continuity requirements, respectively, very analogous to those of classical electromagnetism:

$$\gamma^{-1}\mathbf{\Omega}\mathbb{C}_N(1), \tag{15}$$

$$\mathbf{a} - 2v\hat{\mathbf{v}}\times\mathbf{\Omega} - \gamma^2 v\nabla v\mathbb{C}_T(2). \tag{16}$$

Finally, the scalar shock speed v must satisfy

$$\gamma^2\nabla v + (\mathsf{S} - \mathbf{\Omega}\times\mathsf{I})\cdot\hat{\mathbf{v}}\mathbb{C}_T(2), \tag{17}$$

$$\gamma^2 v' + \gamma\mathbf{a}\cdot\hat{\mathbf{v}} + \gamma v\hat{\mathbf{v}}\cdot\mathsf{S}\cdot\hat{\mathbf{v}}\mathbb{C}. \tag{18}$$

This last scalar condition may also be regarded as a normal continuity condition on $\mathbf{a}$. It may easily be verified that the nine conditions in Eqs. (11), (12), (14), (15), and (16) ensure the uniqueness of the intrinsic metric structure of Σ, as described by the nine Ricci rotation coefficients of the triad ${}_0\Lambda^\mu$, ${}_1\Lambda^\mu$, ${}_3\Lambda^\mu$, while the six conditions in Eqs. (13), (17), and (18) ensure the uniqueness of the six triad components of the second fundamental form of Σ. This interpretation fails for Σ null.

The 14-junction requirements on the Riemann components in Eq. (6) become

$$\gamma^2 v(\rho - \hat{\mathbf{v}}\cdot\mathsf{T}\cdot\hat{\mathbf{v}}) - \gamma^2(1 + v^2)\hat{\mathbf{v}}\cdot\mathbf{t}\mathbb{C}, \tag{19}$$

$$-\gamma^2\hat{\mathbf{v}}\cdot\mathsf{T}\cdot\hat{\mathbf{v}} + \gamma^2 v^2\rho - 2\gamma^2 v\hat{\mathbf{v}}\cdot\mathbf{t}\mathbb{C}, \tag{20}$$

$$\gamma\hat{\mathbf{v}}\cdot\mathsf{T} + \gamma v\mathbf{t}\mathbb{C}_T(2), \tag{21}$$

$$\mathsf{B}\mathbb{C}_N(1), \tag{22}$$

$$\mathsf{A} - \mathsf{T} + \tfrac{1}{3}(\mathrm{Tr}\,\mathsf{T} - 2\rho)\mathsf{I}\mathbb{C}_N(1), \tag{23}$$

$$\gamma(\mathsf{A}\cdot\hat{\mathbf{v}} - v\hat{\mathbf{v}}\times\mathsf{B}\cdot\hat{\mathbf{v}})\mathbb{C}_T(2), \tag{24}$$

$$\gamma(v\mathsf{A}\cdot\hat{\mathbf{v}} - \hat{\mathbf{v}}\times\mathsf{B}\cdot\hat{\mathbf{v}} - v\mathsf{T}\cdot\hat{\mathbf{v}} - \mathbf{t})\mathbb{C}_T(2), \tag{25}$$

$$2\gamma^2 v(2\mathsf{A} + \hat{\mathbf{v}}\cdot\mathsf{A}\cdot\hat{\mathbf{v}}\mathsf{I}) + (2\gamma^2 - 1)(\mathsf{B}\times\hat{\mathbf{v}} - \hat{\mathbf{v}}\times\mathsf{B})\mathbb{C}_T(2), \tag{26}$$

$$(2\gamma^2 - 1)(2\mathsf{A} + \hat{\mathbf{v}}\cdot\mathsf{A}\cdot\hat{\mathbf{v}}\mathsf{I}) + 2v\gamma^2(\mathsf{B}\times\hat{\mathbf{v}} - \hat{\mathbf{v}}\times\mathsf{B}) + 2\mathsf{T} - \mathrm{Tr}\,\mathsf{T}\mathsf{I} + \hat{\mathbf{v}}\cdot\mathsf{T}\cdot\hat{\mathbf{v}}\mathsf{I}\mathbb{C}_T(2). \tag{27}$$

These junction conditions are rich in physical implications when applied to specific cases. One might, for example, have v the same on either side of Σ, $\delta v = 0$, i.e., be using a continuous reference tetrad frame. One then finds the allowed jumps in the dyadic inertial fields from the first set, Eqs. (11)–(18),

$$(\mathsf{I} - \hat{\mathbf{v}}\hat{\mathbf{v}})\cdot\delta\mathsf{S}\cdot(\mathsf{I} - \hat{\mathbf{v}}\hat{\mathbf{v}}) = 0, \tag{28}$$

$$\hat{\mathbf{v}}\times\delta(\mathsf{S} - \mathbf{\Omega}\times\mathsf{I})\cdot\hat{\mathbf{v}} = 0, \tag{29}$$

$$\hat{\mathbf{v}}\times\delta(\mathbf{a} - 2v\hat{\mathbf{v}}\times\mathbf{\Omega}) = 0, \tag{30}$$

$$\hat{\mathbf{v}}\cdot\delta\mathbf{\Omega} = 0, \tag{31}$$

$$\hat{\mathbf{v}}\cdot\delta(\mathbf{a} + v\hat{\mathbf{v}}\cdot\mathsf{S}) = 0. \tag{32}$$

For $\delta\mathsf{S} = 0$ these necessitate $\delta\mathbf{a} = \delta\mathbf{\Omega} = 0$, as one might expect from the analogy of the inertial field to electromagnetism. Similarly, the allowed jumps in the matter and Riemann dyadics, from the second set, Eqs. (19)–(27), must satisfy

$$v\delta[\mathsf{A} - \mathsf{T} + \tfrac{1}{3}(\mathrm{Tr}\,\mathsf{T} - 2\rho)\mathsf{I}] - \hat{\mathbf{v}}\times\delta[\mathsf{B} + \mathbf{t}\times\mathsf{I}] = 0, \tag{33}$$

$$v\delta[\mathsf{B} - \mathbf{t}\times\mathsf{I}] + \hat{\mathbf{v}}\times\delta[\mathsf{A} + \mathsf{T} + \tfrac{1}{3}(\rho - 2\mathrm{Tr}\,\mathsf{T})\mathsf{I}] = 0, \tag{34}$$

(these include the energy–momentum conservation laws $\hat{\mathbf{v}}\cdot\delta\mathsf{T} + v\delta\mathbf{t} = 0$ and $\hat{\mathbf{v}}\cdot\delta\mathbf{t} - v\delta\rho = 0$).

It is, if anything, *more* physical (and convenient), however, to use comoving reference frames on either side of Σ, viz., take $\mathbf{t} = 0$ everywhere in Eqs. (11)–(27); then v on either side is a locally observed normal shock speed. As just one of the results in this case, the four conservation laws in Eqs. (19)–(21) are now seen as relativistic Rankine–Hugoniot relations. If we have a perfect fluid, $\mathsf{T} = -p\mathsf{I}$, they precisely reduce to two conditions given by Taub[4]:

$$\gamma^2 v(p + \rho)\mathbb{C}, \tag{35}$$

$$\gamma^2(p + \rho v^2)\mathbb{C}. \tag{36}$$

If quantities on the two sides of Σ are respectively labeled + and −, an alternative form for these last is

$$\frac{p_+^2 - \rho_+^2 v_+^2}{1 - v_+^2} = \frac{p_-^2 - \rho_-^2 v_-^2}{1 - v_-^2} = \frac{p_+ p_- - \rho_+\rho_- v_+ v_-}{1 - v_+ v_-}. \tag{37}$$

Analogous conditions are involved for the other dyadic quantities. We expect this case of comoving reference frames to be most useful in forming exact solutions.

IV. NULL BOUNDARY

If the bounding 3-surface Σ is null, ${}_2\Lambda^\mu$ is *in* Σ, and is in fact a geodesic null congruence—the bicharacteristics. The transformation matrices to the "standard" orientation are now taken to be

$$^R\Lambda_r = \begin{pmatrix} 2\nu & -2\nu\hat{\mathbf{v}} \\ 0 & \hat{\mathbf{u}} \\ \tfrac{1}{2}\nu^{-1} & \tfrac{1}{2}\nu^{-1}\hat{\mathbf{v}} \\ 0 & \hat{\mathbf{w}} \end{pmatrix}, \quad {}_R\Lambda^r = \begin{pmatrix} \tfrac{1}{4}\nu^{-1} & -\tfrac{1}{4}\nu^{-1}\hat{\mathbf{v}} \\ 0 & \hat{\mathbf{u}} \\ \nu & \nu\hat{\mathbf{v}} \\ 0 & \hat{\mathbf{w}} \end{pmatrix}. \tag{38}$$

This is the most general form for normal boundary motion. With respect to the instantaneous orthonormal axes of either frame, Σ is a 2-surface moving with unit speed in the direction of the normal $\hat{\mathbf{v}}$. Or, Σ appears as an abreast two-dimensional flight of photons, whose world lines are the bicharacteristics. ν is a scalar not fixed by this geometry, but which may be thought of as the observed frequency of the photons, inasmuch as it is only the *ratio* of such ν's on either side of Σ which will appear in the following, and this ratio is precisely the Doppler ratio for the two reference-observer congruences involved.

Equations (9) and (10), satisfied by the normal $\hat{\mathbf{v}}$, are now just

$$\mathbf{\Omega} - \mathbf{\Omega}\cdot\hat{\mathbf{v}}\hat{\mathbf{v}} + \hat{\mathbf{v}} \times [\nu^{-1}\hat{\mathbf{v}}' + \mathsf{S}\cdot\hat{\mathbf{v}} + \mathbf{a}] = 0, \tag{39}$$

$$\hat{\mathbf{v}}\cdot\nabla\times\hat{\mathbf{v}} + 2\hat{\mathbf{v}}\cdot\mathbf{\Omega} = 0. \tag{40}$$

The prime operation now refers to the ${}_2\Lambda^\mu$ congruence; e.g., $\phi' = \nu(\dot\phi + \hat{\mathbf{v}}\cdot\nabla\phi)$, etc. The two sets of junction relations are now

$$\hat{\mathbf{v}}\cdot\nabla\times(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}})\mathbb{C}_T(2), \tag{41}$$

$$(\hat{\mathbf{u}}\hat{\mathbf{w}} + \hat{\mathbf{w}}\hat{\mathbf{u}})'\mathbb{C}_T(1), \tag{42}$$

$$\nu(\nabla\hat{\mathbf{v}} + \hat{\mathbf{v}}\nabla + 2\mathsf{S})\mathbb{C}_T(3), \tag{43}$$

$$\nu^{-1}(\nabla\hat{\mathbf{v}} + \hat{\mathbf{v}}\nabla - 2\mathsf{S})\mathbb{C}_T(3), \tag{44}$$

$$\nu^{-1}\mathbf{\Omega}\mathbb{C}_N(1), \tag{45}$$

$$\mathbf{a} - \hat{\mathbf{v}}\times\mathbf{\Omega} + \mathsf{S}\cdot\hat{\mathbf{v}}\mathbb{C}_T(2), \tag{46}$$

$$\nu^{-1}\nabla\nu + (\mathsf{S} - \mathbf{\Omega}\times\mathsf{I})\cdot\hat{\mathbf{v}}\mathbb{C}_T(2), \tag{47}$$

$$\nu^{-1}\nu' + \nu\mathbf{a}\cdot\hat{\mathbf{v}} + \nu\hat{\mathbf{v}}\cdot\mathsf{S}\cdot\hat{\mathbf{v}}\mathbb{C}, \tag{48}$$

and

$$\rho + \hat{\mathbf{v}}\cdot\mathsf{T}\cdot\hat{\mathbf{v}}\mathbb{C}, \tag{49}$$

$$\hat{\mathbf{v}}\times\mathsf{T} - \mathsf{T}\times\hat{\mathbf{v}}\mathbb{C}_T(2), \tag{50}$$

$$\nu(\mathsf{T}\cdot\hat{\mathbf{v}} + \mathbf{t})\mathbb{C}_T(2), \tag{51}$$

$$\nu^2(\hat{\mathbf{v}}\cdot\mathsf{T}\cdot\hat{\mathbf{v}} - \rho + 2\mathbf{t}\cdot\hat{\mathbf{v}})\mathbb{C}, \tag{52}$$

$$\nu^2(\mathsf{A} + \hat{\mathbf{v}}\times\mathsf{A}\times\hat{\mathbf{v}} - \hat{\mathbf{v}}\times\mathsf{B} + \mathsf{B}\times\hat{\mathbf{v}})\mathbb{C}_T(2), \tag{53}$$

$$\nu(\mathsf{A} - \hat{\mathbf{v}}\times\mathsf{B})\cdot\hat{\mathbf{v}}\mathbb{C}_T(2), \tag{54}$$

$$\nu^{-1}[(\mathsf{A} + \hat{\mathbf{v}}\times\mathsf{B})\cdot\hat{\mathbf{v}} - \mathsf{T}\cdot\hat{\mathbf{v}} + \mathbf{t}]\mathbb{C}_T(2), \tag{55}$$

$$\hat{\mathbf{v}}\cdot\mathsf{A}\cdot\hat{\mathbf{v}} + \tfrac{1}{3}(\rho + \operatorname{Tr}\mathsf{T})\mathbb{C}, \tag{56}$$

$$\hat{\mathbf{v}}\cdot\mathsf{B}\cdot\hat{\mathbf{v}}\mathbb{C}. \tag{57}$$

It is characteristic of the Einstein theory how the last set has reduced in this case of a null boundary. We note that now *six* equations, Eqs. (49)–(52), involve only the stress, momentum density, and energy density T, $\mathbf{t}$, ρ, respectively. A separation of these six conditions into continuity sets for a 3-metric and a second fundamental form is not appropriate.

If we consider a continuous reference frame, so that $\delta\nu = 0$, we find in general that the allowed jumps in the physical components again must satisfy Eqs. (28)–(34); we merely have here the case $v = 1$. If the matter tensor $T_{\mu\nu}$ is characteristic of a vacuum electromagnetic field, the well-known relations $T^{\mu}_{.\mu} = 0$ and $T_\mu^{;\sigma}T_{\sigma\nu} = \frac{1}{4}T_{\alpha\beta}T^{\alpha\beta}g_{\mu\nu}$, which in dyadic notation read

$$\mathsf{T}\cdot\mathsf{T} - \mathbf{t}\mathbf{t} = \tfrac{1}{4}(\rho^2 - 2t^2 + \mathsf{T}:\mathsf{T})\mathsf{I}, \tag{58}$$

$$\mathsf{T}\cdot\mathbf{t} + \rho\mathbf{t} = 0, \tag{59}$$

combine with Eqs. (28)–(34) [$v = 1$] to require that the jumps $\delta\mathsf{A}$ and $\delta\mathsf{B}$ be a type N pure gravitational radiation field with propagation direction $\hat{\mathbf{v}}$, and further, that the jumps $\delta\mathsf{T}$, $\delta\mathbf{t}$, and $\delta\rho$ are in fact a null electromagnetic radiation field with the same propagation vector:

$$\delta\mathsf{T} = -\delta\rho\hat{\mathbf{v}}\hat{\mathbf{v}}, \quad \delta\mathbf{t} = \delta\rho\hat{\mathbf{v}}. \tag{60}$$

This was already shown by Roy and Radhakrishna.[8] It is illuminating also to consider a converse argument; if we require that the jumps $\delta\mathsf{A}$, $\delta\mathsf{B}$ be type N[5]:

$$\delta\mathsf{A} = \hat{\mathbf{v}}\times\delta\mathsf{B}, \quad \delta\mathsf{B} = \delta\mathsf{A}\times\hat{\mathbf{v}}, \tag{61}$$

---

[8] S. R. Roy and L. Radhakrishna, Proc. Roy. Soc. (London) **A275**, 245 (1963).

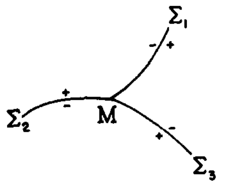


FIG. 1. Three intersecting internal boundaries.

Eqs. (28)–(34) reduce to just Eq. (60). Any field which is to propagate jointly and consistently with a pure gravitational field, must satisfy these in a continuous tetrad frame (covariantly, just $T_{\mu\nu} = \rho\, {}_2\Lambda_\mu\, {}_2\Lambda_\nu$). We repeat for emphasis, however, that the *general* normal jump conditions, Eqs. (49)–(57), with ν discontinuous across Σ, are more likely to be physically and mathematically convenient.

V. INTERSECTING BOUNDARIES

If a space–time is divided into a number of regions, by internal boundaries Σ_1, Σ_2, $\cdots$ etc., the criterion (a) of elemental flatness imposes additional conditions on the transformation matrices ${}_R\Lambda^r$. We now must label these last by boundary: ${}^{1+}_{R}\Lambda^r$, ${}^{2-}_{R}\Lambda^r$, etc.; we further label them by $\pm$ signs to denote on which side of any given boundary is the reference tetrad being rotated into standard position. The additional conditions arise at those points where three or more Σ's meet, and relate the $\boldsymbol{\Lambda}$'s on them.

We illustrate this by considering just one example of the sort of condition to be imposed, that when, say, 3 Σ's intersect in a 2-manifold (cf. Fig. 1), M. Any tetrad, when subjected cyclically as shown to the six transformation matrices on the three Σ's must be finally unmoved, for M to consist of regular points of the space–time manifold. That is, on M we must have (in matrix notation)

$$ {}^{1+}\boldsymbol{\Lambda}\boldsymbol{\eta}\, {}^{1-}\boldsymbol{\Lambda}^T\boldsymbol{\eta}\, {}^{2+}\boldsymbol{\Lambda}\boldsymbol{\eta}\, {}^{2-}\boldsymbol{\Lambda}^T\boldsymbol{\eta}\, {}^{3+}\boldsymbol{\Lambda}\boldsymbol{\eta}\, {}^{3-}\boldsymbol{\Lambda}^T = \boldsymbol{\eta}. \tag{62} $$

For the simplest case, if the $\boldsymbol{\Lambda}$'s are all of the normal form, Eq. (8), and if further the orientation vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ are common to all the Σ's, and so span M, we are just compounding simple Lorentz rotation matrices in the same 2-flat. Equation (62) reduces to a single condition which is a sort of relative velocity law for intersecting shocks:

$$ \frac{1+v_{1+}}{1-v_{1+}}\,\frac{1-v_{1-}}{1+v_{1-}}\,\frac{1+v_{2+}}{1-v_{2+}}\,\frac{1-v_{2-}}{1+v_{2-}}\,\frac{1+v_{3+}}{1-v_{3+}}\,\frac{1-v_{3-}}{1+v_{3-}} = 1. \tag{63} $$

APPENDIX: TANGENTIAL BOUNDARY MOTION

We have also found a need for an explicit dyadic set of junction conditions for the case of tangential boundary motion; seen in space–time, this is when the timelike reference vectors ${}_0\lambda^\mu$ on either side of Σ are normal to ${}_2\Lambda^\mu$, i.e., lie *in* Σ. Such cases arise, e.g., when rotating solutions are to be matched across a boundary which is itself a figure of rotation. As contrasted with Eq. (8), we now take

$$ {}_R\Lambda^r = \begin{array}{cc} & R = \\ \begin{pmatrix} \gamma, & \gamma v \hat{\mathbf{v}} \\ \gamma v, & \gamma \hat{\mathbf{v}} \\ 0, & \hat{\mathbf{w}} \\ 0, & \hat{\mathbf{u}} \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array} \tag{64} $$

$$ r = 0, 1, 2, 3; $$

$\hat{\mathbf{v}}$ is the unit vector giving the direction of the (tangential) relative velocity of the two reference frames. The boundary speed v will without loss of generality be taken numerically equal as seen from either side:

$$ v^- = -v^+; \quad \gamma^- = \gamma^+. \tag{65} $$

$\hat{\mathbf{v}}$ and $\hat{\mathbf{u}}$ are in the instantaneous boundary 2-surface; $\hat{\mathbf{w}}$ is the normal to the boundary 2-surface, so

$$ \hat{\mathbf{w}} \cdot (\boldsymbol{\nabla} \times \hat{\mathbf{w}}) = 0, \quad \hat{\mathbf{w}} \times [\dot{\hat{\mathbf{w}}} + \mathsf{S}^\star \cdot \hat{\mathbf{w}}] = 0. \tag{66} $$

The 18 conditions of Eq. (5) are found to be

$$ \hat{\mathbf{u}} \cdot \boldsymbol{\nabla}\hat{\mathbf{w}} \cdot \hat{\mathbf{u}}\mathbb{C}, \tag{67} $$

$$ [\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}\hat{\mathbf{w}} + v(\dot{\hat{\mathbf{w}}} + \boldsymbol{\omega} \times \hat{\mathbf{w}})] \cdot \hat{\mathbf{u}}\mathbb{C}, \tag{68} $$

$$ [v\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}\hat{\mathbf{w}} + \dot{\hat{\mathbf{w}}} + \boldsymbol{\omega} \times \hat{\mathbf{w}}] \cdot \hat{\mathbf{u}}\mathbb{C}, \tag{69} $$

$$ v(\gamma^2\dot{v} + \mathbf{a} \cdot \hat{\mathbf{v}}) + \gamma^2\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}v + \hat{\mathbf{v}} \cdot \mathsf{S} \cdot \hat{\mathbf{v}}\mathbb{C}, \tag{70} $$

$$ \gamma^2\dot{v} + \mathbf{a} \cdot \hat{\mathbf{v}} + v(\gamma^2\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}v + \hat{\mathbf{v}} \cdot \mathsf{S} \cdot \hat{\mathbf{v}})\mathbb{C}, \tag{71} $$

$$ \hat{\mathbf{u}} \cdot [\gamma^2\boldsymbol{\nabla}v + \hat{\mathbf{v}} \cdot \mathsf{S} - \boldsymbol{\Omega} \times \hat{\mathbf{v}}]\mathbb{C}, \tag{72} $$

$$ (\dot{\hat{\mathbf{v}}} - \hat{\mathbf{v}} \cdot \mathsf{S}^\star) \times \hat{\mathbf{v}}\mathbb{C}(2), \tag{73} $$

$$ (\hat{\mathbf{v}} \cdot \boldsymbol{\nabla}\hat{\mathbf{v}} - \mathbf{a}) \times \hat{\mathbf{v}}\mathbb{C}(2), \tag{74} $$

$$ \hat{\mathbf{w}} \times \{\boldsymbol{\nabla}\hat{\mathbf{v}} + v(\mathsf{S} - \boldsymbol{\Omega} \times \mathsf{I}) + v\hat{\mathbf{v}}(\dot{\hat{\mathbf{v}}} + \boldsymbol{\omega} \times \hat{\mathbf{v}}) + v^2\hat{\mathbf{v}}\mathbf{a}\} \times \hat{\mathbf{v}}\mathbb{C}(4), \tag{75} $$

$$ \hat{\mathbf{w}} \times \{v\boldsymbol{\nabla}\hat{\mathbf{v}} + (\mathsf{S} - \boldsymbol{\Omega} \times \mathsf{I}) + v^2\hat{\mathbf{v}}(\dot{\hat{\mathbf{v}}} + \boldsymbol{\omega} \times \hat{\mathbf{v}}) + v\hat{\mathbf{v}}\mathbf{a}\} \times \hat{\mathbf{v}}\mathbb{C}(4). \tag{76} $$

The 14 conditions of Eq. (6) are

$$ \hat{\mathbf{v}} \cdot \mathsf{B} \cdot \hat{\mathbf{v}}\mathbb{C}, \tag{77} $$

$$ \hat{\mathbf{u}} \cdot \mathsf{Q} \cdot \hat{\mathbf{u}} + \hat{\mathbf{w}} \cdot \mathsf{P} \cdot \hat{\mathbf{w}}\mathbb{C}, \tag{78} $$

$$ \hat{\mathbf{v}} \cdot \mathsf{Q} \cdot \hat{\mathbf{v}}\mathbb{C}, \tag{79} $$

$$ \hat{\mathbf{u}} \cdot (\mathsf{P} - \mathsf{Q}) \cdot \hat{\mathbf{w}}\mathbb{C}, \tag{80} $$

$$\hat{\mathbf{u}} \cdot [v\mathsf{B}^{\star T} + \hat{\mathbf{v}} \times \mathsf{P}] \cdot \hat{\mathbf{v}}\mathbb{C}, \tag{81}$$

$$\hat{\mathbf{u}} \cdot [\mathsf{B}^{\star T} + v\hat{\mathbf{v}} \times \mathsf{P}] \cdot \hat{\mathbf{v}}\mathbb{C}, \tag{82}$$

$$\hat{\mathbf{v}} \times [\mathsf{Q} - v\hat{\mathbf{v}} \times \mathsf{B}^\star] \cdot \hat{\mathbf{v}}\mathbb{C}(2), \tag{83}$$

$$\hat{\mathbf{v}} \times [v\mathsf{Q} - \hat{\mathbf{v}} \times \mathsf{B}^\star] \cdot \hat{\mathbf{v}}\mathbb{C}(2), \tag{84}$$

$$\hat{\mathbf{v}} \times \{v[\mathsf{B}^{\star T} + v\hat{\mathbf{v}} \times \mathsf{P}] - [\mathsf{Q} - v\hat{\mathbf{v}} \times \mathsf{B}^\star] \times \hat{\mathbf{v}}\} \cdot \hat{\mathbf{w}}\mathbb{C}(2), \tag{85}$$

$$\hat{\mathbf{v}} \times \{[\mathsf{B}^{\star T} + v\hat{\mathbf{v}} \times \mathsf{P}] - v[\mathsf{Q} - v\hat{\mathbf{v}} \times \mathsf{B}^\star] \times \hat{\mathbf{v}}\} \cdot \hat{\mathbf{w}}\mathbb{C}(2). \tag{86}$$

We have in these last used the dyadics P and Q, $\mathsf{B}^\star$ and $\mathsf{S}^\star$:

$$\begin{gathered} \mathsf{P} = \mathsf{A} - \mathsf{T} + \tfrac{1}{3}(\mathrm{Tr}\,\mathsf{T} - 2\rho)\mathsf{I}, \\ \mathsf{Q} = \mathsf{A} + \mathsf{T} + \tfrac{1}{3}(\rho - 2\,\mathrm{Tr}\,\mathsf{T})\mathsf{I}, \\ \mathsf{B}^\star = \mathsf{B} - \mathbf{t} \times \mathsf{I}, \quad \mathsf{S}^\star = \mathsf{S} - (\boldsymbol{\Omega} - \boldsymbol{\omega}) \times \mathsf{I}. \end{gathered} \tag{87}$$

The energy-momentum conservation laws contained in the above are

$$\hat{\mathbf{v}} \times \mathsf{T} \cdot \hat{\mathbf{w}}\mathbb{C}(2), \tag{88}$$

$$(\mathbf{t} + v\hat{\mathbf{v}} \cdot \mathsf{T}) \cdot \hat{\mathbf{w}}\mathbb{C}, \tag{89}$$

$$(v\mathbf{t} + \hat{\mathbf{v}} \cdot \mathsf{T}) \cdot \hat{\mathbf{w}}\mathbb{C}. \tag{90}$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Analyticity in the Potential Strength*

D. MASSON
Department of Mathematics, University of Toronto, Toronto, Canada

(Received 25 May 1967)

The partial-wave S-matrix element is examined as a function of potential strength g. It is shown to be related to a series of Stieltjes in g. This is accomplished by looking at the zeros of the Jost function below threshold and the zeros and poles of $\tan \delta_l$ above threshold. The series of Stieltjes property is used to establish the convergence of the Padé approximant method of summing the Born series.

I. INTRODUCTION

THE subject of potential scattering has proved to be a useful testing ground for theoretical ideas. In recent times, the investigations were stimulated by the dispersion relation approach to scattering problems, where the analytic properties of the scattering amplitude as a function of energy, momentum transfer, and angular momentum are of prime importance.[1]

Dispersion relation methods were principally developed to cope with the problem of strong interactions, where perturbation methods were inadequate. A new and promising approach to the problem of strong interactions consists of an approximate summation of the perturbation expansion using the Padé approximant.[2,3] The method is intimately linked with the analytic properties in the strength g of the interaction and is the motivation for our present investigation.

One must be cautioned, however, in drawing analogies between perturbation expansions in field theory and potential scattering. Although one may feel confident that bound states and resonances reflect singularities in the analytically continued perturbation expansion, in field theory the expansion itself may be asymptotic.[4] This suggests that in field theory there may be additional singularities such as a branch point at the origin ($g = 0$).[5] The Padé, however, may still be capable of summing the series.[6] A more serious situation is the presence of an essential singularity at the origin,[7] which makes it impossible to construct the scattering amplitude from its perturbation expansion.

* Supported by the National Research Council of Canada.

[1] See, for example, R. Blankenbecler, M. L. Goldberger, N. N Khuri, and S. B. Treiman, Ann. Phys. **10**, 62 (1960). For a recent review, see V. de Alfaro and T. Regge, *Potential Scattering* (North-Holland Publishing Company, Amsterdam, 1965).

[2] For a review of the Padé and some of its applications, see G. A. Baker, Jr., *Advances in Theoretical Physics*, K. A. Brueckner, Ed. (Academic Press Inc., New York, 1965) p. 1, Vol. I.

[3] Some recent applications of the Padé are: J. L. Gammel and F. A. McDonald, Phys. Rev. **142**, 1245 (1966); D. Masson, J. Math. Phys. **8**, 512 (1967); J. Nuttall, Phys. Rev. **157**, 1312 (1967).

[4] For recent work connected with the conjecture of F. J. Dyson, Phys. Rev. **85**, 631 (1952) and a list of related references see W. M. Frank, J. Math. Phys. **5**, 363 (1964).

[5] A. Peres, J. Math. Phys. **4**, 332 (1963).

[6] G. A. Baker, Jr., and R. Chisholm, J. Math. Phys. **11**, 1900 (1966).

[7] C. S. Lam, Nuovo Cimento **47A**, 451 (1967); **50A**, 504 (1967).

$$\hat{\mathbf{u}} \cdot [v\mathsf{B}^{\star T} + \hat{\mathbf{v}} \times \mathsf{P}] \cdot \hat{\mathbf{v}}\mathbb{C}, \tag{81}$$

$$\hat{\mathbf{u}} \cdot [\mathsf{B}^{\star T} + v\hat{\mathbf{v}} \times \mathsf{P}] \cdot \hat{\mathbf{v}}\mathbb{C}, \tag{82}$$

$$\hat{\mathbf{v}} \times [\mathsf{Q} - v\hat{\mathbf{v}} \times \mathsf{B}^{\star}] \cdot \hat{\mathbf{v}}\mathbb{C}(2), \tag{83}$$

$$\hat{\mathbf{v}} \times [v\mathsf{Q} - \hat{\mathbf{v}} \times \mathsf{B}^{\star}] \cdot \hat{\mathbf{v}}\mathbb{C}(2), \tag{84}$$

$$\hat{\mathbf{v}} \times \{v[\mathsf{B}^{\star T} + v\hat{\mathbf{v}} \times \mathsf{P}] - [\mathsf{Q} - v\hat{\mathbf{v}} \times \mathsf{B}^{\star}] \times \hat{\mathbf{v}}\} \cdot \hat{\mathbf{w}}\mathbb{C}(2), \tag{85}$$

$$\hat{\mathbf{v}} \times \{[\mathsf{B}^{\star T} + v\hat{\mathbf{v}} \times \mathsf{P}] - v[\mathsf{Q} - v\hat{\mathbf{v}} \times \mathsf{B}^{\star}] \times \hat{\mathbf{v}}\} \cdot \hat{\mathbf{w}}\mathbb{C}(2). \tag{86}$$

We have in these last used the dyadics P and Q, $\mathsf{B}^\star$ and $\mathsf{S}^\star$:

$$\begin{aligned} \mathsf{P} &= \mathsf{A} - \mathsf{T} + \tfrac{1}{3}(\mathrm{Tr}\,\mathsf{T} - 2\rho)\mathsf{I}, \\ \mathsf{Q} &= \mathsf{A} + \mathsf{T} + \tfrac{1}{3}(\rho - 2\,\mathrm{Tr}\,\mathsf{T})\mathsf{I}, \\ \mathsf{B}^\star &= \mathsf{B} - \mathbf{t} \times \mathsf{I}, \quad \mathsf{S}^\star = \mathsf{S} - (\mathbf{\Omega} - \boldsymbol{\omega}) \times \mathsf{I}. \end{aligned} \tag{87}$$

The energy-momentum conservation laws contained in the above are

$$\hat{\mathbf{v}} \times \mathsf{T} \cdot \hat{\mathbf{w}}\mathbb{C}(2), \tag{88}$$

$$(\mathbf{t} + v\hat{\mathbf{v}} \cdot \mathsf{T}) \cdot \hat{\mathbf{w}}\mathbb{C}, \tag{89}$$

$$(v\mathbf{t} + \hat{\mathbf{v}} \cdot \mathsf{T}) \cdot \hat{\mathbf{w}}\mathbb{C}. \tag{90}$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 8, NUMBER 11 NOVEMBER 1967

Analyticity in the Potential Strength*

D. MASSON
Department of Mathematics, University of Toronto, Toronto, Canada

(Received 25 May 1967)

The partial-wave S-matrix element is examined as a function of potential strength g. It is shown to be related to a series of Stieltjes in g. This is accomplished by looking at the zeros of the Jost function below threshold and the zeros and poles of tan δ_l above threshold. The series of Stieltjes property is used to establish the convergence of the Padé approximant method of summing the Born series.

I. INTRODUCTION

THE subject of potential scattering has proved to be a useful testing ground for theoretical ideas. In recent times, the investigations were stimulated by the dispersion relation approach to scattering problems, where the analytic properties of the scattering amplitude as a function of energy, momentum transfer, and angular momentum are of prime importance.[1]

Dispersion relation methods were principally developed to cope with the problem of strong interactions, where perturbation methods were inadequate. A new and promising approach to the problem of strong interactions consists of an approximate summation of the perturbation expansion using the Padé approximant.[2,3] The method is intimately linked with the analytic properties in the strength g of the interaction and is the motivation for our present investigation.

One must be cautioned, however, in drawing analogies between perturbation expansions in field theory and potential scattering. Although one may feel confident that bound states and resonances reflect singularities in the analytically continued perturbation expansion, in field theory the expansion itself may be asymptotic.[4] This suggests that in field theory there may be additional singularities such as a branch point at the origin ($g = 0$).[5] The Padé, however, may still be capable of summing the series.[6] A more serious situation is the presence of an essential singularity at the origin,[7] which makes it impossible to construct the scattering amplitude from its perturbation expansion.

* Supported by the National Research Council of Canada.

[1] See, for example, R. Blankenbecler, M. L. Goldberger, N. N Khuri, and S. B. Treiman, Ann. Phys. **10**, 62 (1960). For a recent review, see V. de Alfaro and T. Regge, *Potential Scattering* (North-Holland Publishing Company, Amsterdam, 1965).

[2] For a review of the Padé and some of its applications, see G. A. Baker, Jr., *Advances in Theoretical Physics*, K. A. Brueckner, Ed. (Academic Press Inc., New York, 1965) p. 1, Vol. I.

[3] Some recent applications of the Padé are: J. L. Gammel and F. A. McDonald, Phys. Rev. **142,** 1245 (1966); D. Masson, J. Math. Phys. **8,** 512 (1967); J. Nuttall, Phys. Rev. **157**, 1312 (1967).

[4] For recent work connected with the conjecture of F. J. Dyson, Phys. Rev. **85,** 631 (1952) and a list of related references see W. M. Frank, J. Math. Phys. **5,** 363 (1964).

[5] A. Peres, J. Math. Phys. **4,** 332 (1963).

[6] G. A. Baker, Jr., and R. Chisholm, J. Math. Phys. **11,** 1900 (1966).

[7] C. S. Lam, Nuovo Cimento **47A,** 451 (1967); **50A,** 504 (1967).

Here we attack the potential problem, not by using the integral equation[2,8] for the scattering amplitude, but through the radial Schrödinger equation and the Jost function.

The analytic properties of the Jost function in the potential strength g are particularly simple. For potentials which are less singular than r^{-2} for small r and which decrease faster than r^{-1} for large r, the Jost function is an entire function of g. For potentials which satisfy

$$\left| \int_0^\infty [V(r)]^{\frac{1}{2}}\, dr \right| < \infty,$$

we show that the Jost function is an entire function of order $\frac{1}{2}$ and finite type. For potentials which decrease exponentially or faster, we examine the distribution of poles and zeros of the partial-wave S-matrix element, and show that for energy $E < 0$, it is related to a series of Stieltjes[9] in g. This property is of interest, since it is a sufficient condition for the convergence of the Padé method.[2]

For $E > 0$, it is convenient to deal with the zeros and poles of $\tan \delta_l$. If the potential has no zeros for $0 < r < \infty$, we find that $\tan \delta_l$ is related to a series of Stieltjes.

In Sec. II, we review the basic notations. In Sec. III, we examine the distribution of zeros of the Jost function $f_l(k, g)$ for pure imaginary $k (E < 0)$. We do a similar analysis for the zeros and poles of $\tan \delta_l(k, g)$ for real $k (E > 0)$ in Sec. IV. In Sec. V, we establish the order and type of $f_l(k, g)$ as a function of g.

II. PRELIMINARIES

The radial Schrödinger equation is (units $\hbar^2/2m = 1$)

$$\frac{d^2}{dr^2} y_l(k, g, r) + \left[k^2 - \frac{l(l+1)}{r^2} - gV(r)\right] y_l(k, g, r) = 0, \quad (1)$$

where the energy $E = k^2$. The regular solution $\phi_l(k, g, r)$ is defined by the boundary condition (the potential is assumed to be less singular than r^{-2} at the origin),

$$\lim_{r\to 0} r^{-l-1}\phi_l(k^2, g, r) = 1. \quad (2)$$

The Jost solution $f_l(k, g, r)$ satisfies the boundary condition (the potential is assumed to fall off faster than r^{-1}),

$$\lim_{r\to\infty} e^{ikr} f_l(k, g, r) = 1. \quad (3)$$

The Jost function is defined by

$$f_l(k, g) = W_r[f_l(k, g, r), \phi_l(k^2, g, r)], \quad (4)$$

where

$$W_r(\alpha, \beta) = \alpha \frac{d\beta}{dr} - \beta \frac{d\alpha}{dr}.$$

The asymptotic form of ϕ_l for large r is then

$$\phi_l(k^2, g, r) \sim [f_l(k, g)e^{ikr} - f_l(-k, g)e^{-ikr}]/2ik. \quad (5)$$

The S-matrix element $S_l(k, g) = \exp[2i\delta_l(k, g)]$ is given in terms of the Jost function by

$$S_l(k, g) = (-1)^l f_l(k, g)/f_l(-k, g). \quad (6)$$

Two properties are immediate:

A. The Jost function is an entire function of g. This follows from the fact that because the boundary conditions (2) and (3) are independent of g, both solutions f_l and ϕ_l are entire functions of g (Poincaré's theorem).[1]

B. The Jost functions $f_l(k, g)$ and $f_l(-k, g)$ cannot both vanish for the same value of $k \neq 0$ and g. This follows from the fact that if they did, ϕ_l [from Eq. (5)] would vanish identically, contradicting Eq. (2).

We occasionally refer to functions which we call an "ordinary" series of Stieltjes or an "extended" series of Stieltjes. By this we mean the following. If $f(z) = \int d\phi(u)(1 - uz)^{-1}$ has a power series expansion $f(z) = \sum f_n z^n$, and $\phi(u)$ is a bounded monotonic function taking on infinitely many values in the interval of integration, we call $f(z)$ an ordinary series of Stieltjes if the interval is contained in half of the real axis and an extended series of Stieltjes otherwise.

For example, if $f(z)$ is a meromorphic function with infinitely many real poles z_n with residues R_n having all R_n/z_n of the same sign and if $f(z) = \sum R_n(z - z_n)^{-1}$ converges, then $f(z)$ is an ordinary series of Stieltjes if the z_n are either all positive or negative and an extended series of Stieltjes if the z_n are both positive and negative.

The series of Stieltjes property is important for the application of the Padé approximant, where the $[N, M]$ Padé is defined to be a ratio of polynomials $P_{N,M}(z)/Q_{N,M}(z)$—$P_{N,M}$ and $Q_{N,M}$ have degree M and N, respectively—having the same power series expansion as $f(z)$ up to and including the term z^{N+M}. If $f(z)$ is a series of Stieltjes, then the Padé may be used to construct the function from its power series expansion. For our present purposes, the following theorem suffices.[2]

Theorem: If $[f(z) - f_0 - f_1 z \cdots - f_{s-1}z^{s-1}]/z^s$ is a series of Stieltjes with finite radius of convergence,

[8] S. Tani, Phys. Rev. **139**, B 1011 (1965).

[9] Within the text we use a broader definition of the term "series of Stieltjes" than is usual. The standard definition of a series of Stieltjes (see Ref. 2) is essentially what we call an "ordinary" series of Stieltjes.

then the $\lim N \to \infty$ $[N, N + j + s]$ converges to $f(z)$ for $j = -1, 1, 3, \cdots$, and z not a singular point of the function. If $f(z)$ is an ordinary series of Stieltjes, one may also include $j = 0, +2, +4, \cdots$.

III. BOUND STATES

We consider here imaginary $k(E < 0)$ and first restrict ourselves to positive semi-definite Yukawa type potentials. That is to say $V(r) \geq 0$ and $V(r) = \int_\mu^\infty e^{-\alpha r}\sigma(\alpha)\,d\alpha$, $\mu > 0$. A familiarity with the analytic properties of the Jost function in the variable k will be assumed. Since we deal only with fixed angular momentum l, we will in what follows suppress the angular momentum subscripts of Sec. II.

In Sec. V, we show that $f(k, g)$ is an entire function of g of order $\frac{1}{2}$ and type $\int_0^\infty (V(r))^{\frac{1}{2}}\,dr$. Using these results, we may thus write $f(k, g)$ as an infinite product:

$$f(k, g) = f(k, 0) \prod_n [1 - g/g_n(k)], \tag{7}$$

where

$$f(k, 0) = 2\pi^{-\frac{1}{2}}\Gamma(l + \tfrac{3}{2})(2/ik)^l.$$

The S-matrix element from Eqs. (6) and (7) is then

$$S(k, g) = \prod_n \frac{1 - g/g_n(k)}{1 - g/g_n(-k)}, \tag{8}$$

which we may rewrite, apart from subtractions, in a Mittag–Leffler expansion

$$S(k, g) = \sum_n - \frac{R_n(k)/g_n(-k)}{1 - g/g_n(-k)}. \tag{9}$$

The expression (9) is only a formal expansion and does not converge. The convergence may be improved by performing s subtractions so that Eq. (9) would read as

$$S(k, g) = 1 + S_1(k)g + \cdots S_{s-1}(k)g^{s-1} - g^s \times \sum \frac{R_n(k)/g_n^{s+1}(-k)}{1 - g/g_n(-k)}. \tag{10}$$

A sufficient condition for (10) to be a valid expansion is[10]

$$\text{for} \lim_{|g| \to \infty} |S(k, g)|g^{-s} = 0$$

for all $\arg g \neq \theta$, where

$$\lim_{n \to \infty} \arg g_n = \theta.$$

We assume that Eq. (9) is valid apart from a finite number of subtractions. That at least one subtraction is necessary can be seen from the fact that for k and g real, $S = \exp(2i\delta)$ where δ is real. Also, for even l, we show that $R_n(k)/g_n(-k)$ is positive for $k = iK$, $K > 0$ so that Eq. (9) would contradict the fact that $S(k, 0) = 1$. If we ignore the problem of subtractions, we are able to prove the following theorem.

Theorem: For $-\mu^2/4 < E < 0$, $[S(k, g) - 1]/g$ is an ordinary series of Stieltjes in g.

To prove the theorem, we show that for $k = iK$, $0 < K < \mu/2$, the poles $g_n(-k)$ are real and negative and that the residues $R_n(k)$ are all positive or negative depending on whether l is odd or even. The poles $g_n(-k)$ are of course associated with the bound states (when $g = g_n(-K)$, $K > 0$; the potential $gV(r)$ has a bound state with energy $E = -K^2$). We first establish the following lemmas.

Lemma 1: $g_n(-k)$ is real for $k = iK$, $K \geq 0$. Consider $k = iK$, $K \geq 0$, and $g = g_n(-k)$. Writing Eq. (1) for ϕ and ϕ^*, we have the identity

$$\frac{d}{dr} W_r(\phi, \phi^*) + [g_n(-k) - g_n^*(-k)]V(r)\,|\phi|^2 = 0. \tag{11}$$

From the asymptotic form (5) and the boundary condition (2), we have, on integrating (11),

$$[g_n(-k) - g_n^*(-k)]\int_0^\infty V(r)\,|\phi|^2\,dr = 0.$$

Since $\int_0^\infty V(r)\,|\phi|^2 \neq 0$, we must have

$$g_n(-k) - g_n^*(-k) = 0.$$

Lemma 2: $g_n(-k) < 0$ for $k = iK$, $K \geq 0$. In Eq. (1) put $y = \phi$, $k = iK$, and $g = g_n(-k)$. Multiplying by ϕ and integrating, one has the identity

$$g_n(-k)\int V(r)\phi^2\,dr = \phi\frac{d\phi}{dr} - \int dr\left[\left(\frac{d\phi}{dr}\right)^2 + \left(K^2 + l\frac{(l+1)}{r^2}\right)\phi^2\right].$$

If $K \geq 0$, one can take the limits 0 to ∞ to get

$$g_n(-k) = -\int_0^\infty dr\left[\left(\frac{d\phi}{dr}\right)^2 + \left(K^2 + l\frac{(l+1)}{r^2}\right)\phi^2\right]\Big/\int_0^\infty V(r)\phi^2\,dr < 0. \tag{12}$$

Lemma 3: $\dfrac{dg_n(-iK)}{dK} < 0$ for $K > 0$. In Eq. (1) put $y = \phi$, $k = iK$ and $g = g_n(-iK)$. Denoting

[10] The proof is a standard application of Cauchys theorem. For example, see A. I. Markuschevick, *Theory of Functions of a Complex Variable* (Prentice-Hall, Inc., Englewood Cliffs, N.J. 1965), Vol. II, Theorem 2.7.

$d\phi[-K^2, g_n(-iK), r]/dK$ by ϕ', one has the identity

$$\frac{d}{dr} W_r(\phi, \phi') + 2K\phi^2 + \frac{dg_n(-iK)}{dK} V(r)\phi^2 = 0.$$

Integrating from 0 to ∞ for $K > 0$, one has

$$\frac{dg_n(-iK)}{dK} = -2K\int_0^\infty \phi^2 .dr \Big/ \int_0^\infty V(r)\phi^2\, dr < 0. \quad (13)$$

Lemma 4: $f(-iK, g)$ has no multiple zeros for $K \geq 0$. Suppose $g_n(-iK)$ and $g_p(-iK)$ were coincident. Now ϕ and $\dot\phi = d\phi/dg$ satisfy the identity

$$\frac{d}{dr} W_r(\phi, \dot\phi) - V(r)\phi^2 = 0. \quad (14)$$

If one puts $g = g_n(-iK)$, $K > 0$ and integrates, this becomes $\int_0^\infty dr\, \phi^2 = 0$, which is a contradiction. The point $K = 0$ can easily be included.

Lemma 5: $g_n(-iK)$ is an analytic function of K in the neighborhood of $K = 0$. Now

$$h(K, g) = K^l f(-iK, g)$$

is an analytic function of g and K in the neighborhood of $K = 0$.[1] Since $dh/dg \neq 0$ at the point $K = 0$, $g = g_n$, it follows that $g_n(-iK)$ is analytic in the same neighborhood (implicit function theorem, see Ref. 10, Theorem 3.11).

Lemma 6: The limit as $K \to 0$ of $K^{-2l-1}[g_n(-iK) - g_n(iK)] = (-1)^l C_n$, where $C_n < 0$. For $l = 0$ this is just a restatement of Lemma 3 for $K \to 0$. For general l, we proceed as follows. Since from Lemma 5, $g_n(-k)$ is analytic in the neighborhood of $k = 0$, we can consider the limit as $k \to 0$ for k real. We rewrite identity (11) for $g = g_n(-k)$ with k real noting that $g_n(-k)$ is now complex, but since it is a real analytic function of K in the neighborhood of $K = 0$, we have $g_n^*(-k) = g_n(k)$. Integrating (11) we get

$$[g_n(-k) - g_n(k)]\int_0^\infty V(r)\,|\phi|^2 = |f(k, g_n(-ik))|^2 \frac{ik}{2k^2}.$$

For small k we put $g_n(-k) - g_n(k) = -C_n i k^{2r+1}$. Since

$$|f(k, g_n(-k))|^2 \sim \prod_{m \neq n} [1 - (g_n(0)/g_m(0))]^2 \times C_n^2 2^{2l+2}\Gamma^2(l + \tfrac{3}{2})k^{4r+2-2l}/g_n^2(0)\pi,$$

we must have $r = l$ and $C_n < 0$.

We are now in a position to prove the main theorem by considering the zeros and poles of S in the variable

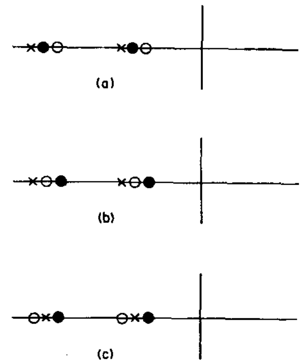


FIG. 1. The zeros 0 and poles X of S in the g plane for $E \lessdot 0$ and $V(r) \geq 0$ for: (a) $l = 0$, (b) l even ($l \neq 0$), and (c) l odd. The solid black points are their $E = 0$ positions.

g. For $k = 0$ the zeros and poles coincide and S has the value 1. For $k = -iK$, K small, real, and positive, the poles $g_n(-iK)$ and zeros $g_n(iK)$ are distributed (from Lemmas 1–6) as shown in Fig. 1.

For $0 < K < \mu/2$ the zeros and poles of S must remain distributed in this manner except for:

A. additional zeros which may appear in complex pairs at infinity, move into the finite g plane and possibly become real at some value of K,

B. additional real zeros on the positive real g axis which may appear from $+\infty$ at certain values of K.

In either case one sees that near a pole $g_n(-iK)$, $S \approx R_n(iK)/[g - g_n(-iK)]$, where the residues $R_n(iK)$ are negative for l even and positive for l odd.

From the discussion in Sec. II on series of Stieltjes and if expansion (10) is valid with one subtraction, we see that $(S - 1)/g$ is an ordinary series of Stieltjes. If s subtractions are required, we see that $(S - 1 - S_1 g \cdots - S_{t-1}g^{t-1})/g^t$ is an ordinary series of Stieltjes for $t \geq s$.

From the theorems on the Padé, one may then conclude the following corollary.

Corollary: The $[N, N + j + s]$ Padé approximant converges to S for large N, $j \geq -1$, $-(\mu^2/4) < E < 0$, and $g \neq g_n(-iK)$.

One may note that the zeros of the type appearing in A. and B. and the number of subtractions are related. For example, if only one subtraction is necessary, then by considering Eq. (10) for g complex, we have $\operatorname{Im} S = -\operatorname{Im} g \sum R_n/|g - g_n|^2$ which cannot vanish for $\operatorname{Im} g \neq 0$, and hence S has no complex zeros. Also, for real g one would have

$$\frac{dS}{dg} = \sum - \frac{R_n/g_n^2}{(1 - g/g_n)^2}$$

which is always positive for l even, and hence there can be no zeros of type B. For l odd, $dS/dg < 0$ and only one zero of type B. may occur.

The case of one subtraction appears optimum for the convergence of the Padé to the partial-wave scattering amplitude $T = (S-1)/2i$. In this case one has both T/g and g/T as ordinary series of Stieltjes and the $[N, N+j]$ Padé converges to T for $j = 0, \pm 1, \pm 2, \cdots$.

One may be curious as to what happens for $K \geq \mu/2$. Since $f(iK, g)$ is singular and infinite at $K = n\,\mu/2$, $n = 1, 2, \cdots$, the behavior of the zeros is complicated. If the singularity is a branch point, its effect is to make the zeros of $f(iK, g)$ singular. Since the singularity is known to occur only in $S_1(k)$, one is forced to write S in at least a twice subtracted form. What happens is that as $K \to \mu/2$ one of the zeros approaches the origin while all the other zeros match up with the poles. For $\mu/2 < K < \mu$, the $g_n(iK)$ are generally complex. If the singularity in K is a simple pole, as it is for the exponential or Hulthen potentials for $l = 0$, then as $K \to \mu/2$, $g_1(iK) \to 0$ and $g_n(iK) \to g_{n-1}(-iK)$. For $\mu > K > \mu/2$ one has $g_1(iK) < 0$ and $g_{n-1}(-iK) < g_n(iK) < g_{n-2}(-iK)$. The behavior at the other singular points is similar.

In spite of this peculiar behavior at $K = n\,\mu/2$, the residues $R_n(iK)$ cannot change sign and S is still related to an ordinary series of Stieltjes except that one is forced to write S in at least an $n+1$ subtracted form for $(n+1)\,\mu/2 < K < n\,\mu/2$.

Let us now relax the condition on the positive definiteness of the potential. It will again follow that $f(k, g)$ is an entire function of order $\frac{1}{2}$, although in a derivation similar to that in Sec. V, one must take into account turning points. For example, if $V(r)$ had a single zero at $r = a$ and $V(r) > 0$ for $r < a$, one could show that $f(k, g)$ was an entire function of order $\frac{1}{2}$ and type

$$\left\{\left[\int_0^a V^{\frac{1}{2}}(r)\,dr\right]^2 + \left[\int_a^\infty [-V(r)]^{\frac{1}{2}}\,dr\right]^2\right\}^{\frac{1}{2}}.$$

Also, although $\int_0^\infty V(r)\phi^2\,dr$ is no longer positive definite, Lemmas 1, 4, and 5 remain valid. This is because of the identity $[g = g_n(-k)$, $k = iK$, and $K \geq 0]$,

$$g_n(-iK)\int_0^\infty V(r)\,|\phi|^2\,dr = -\int_0^\infty dr\left(\left|\frac{d\phi}{dr}\right|^2 + \left[K^2 + l\,\frac{(l+1)}{r^2}\right]|\phi|^2\right) \quad (15)$$

which tells us that $g_n(-iK)$ is real (hence also ϕ) and that $\int_0^\infty V(r)\phi^2\,dr \neq 0$ and has the opposite sign to $g_n(-iK)$. Although Lemmas 2 and 3 are not valid, they

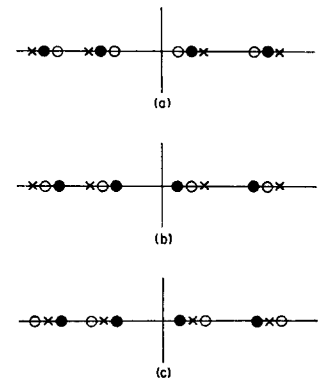


FIG. 2. The same as Fig. 1 but for $V(r)$ with zeros.

may be combined to state that for $K > 0$

$$\frac{1}{g_n(-iK)}\cdot\frac{dg_n(-iK)}{dK} = 2K\int_0^\infty \phi^2\,dr\int_0^\infty dr \times\left[\left(\frac{d\phi}{dr}\right)^2 + \left(K^2 + l\,\frac{(l+1)}{r^2}\right)\phi^2\right]. \quad (16)$$

Also, Lemma 6 must be restated to say that C_n has the same sign as $g_n(-iK)$.

A similar analysis for the distribution of zeros and poles of S results, except they now extend over the whole real axis as in Fig. 2. As a result, we have $R_n/g_n > 0$ for l even and <0 for l odd. Hence (assuming only one subtraction), $(S - 1 - gS_1)/g^2$ and $1/(S-1) - (1/gS_1)$ are extended series of Stieltjes and the $[N, N+j]$ Padé approximant for the partial-wave scattering amplitude T converges to T for large N, $j = \pm 1, \pm 3, \cdots$, and $-\mu^2/4 < E < 0$. If s subtractions are required, one may only assert that for large N the $[N, N+j+s]$ Padé for S converges to S for $j = -1, 1, 3, \cdots$, and $-\mu^2/4 < E < 0$.

For potentials which decrease faster than an exponential (for example potentials of finite range) all that has been said remains valid if one replaces μ by $+\infty$.

IV. SCATTERING

In the previous section, we were able to establish, by examining the zeros and poles of the partial-wave S matrix, the convergence of the Padé method for negative energies. For positive energies, the zeros and poles of S are complex. Since our proof depended on the series of Stieltjes property of S, it cannot be applied here. One would suspect, however, that the Padé approximant would also converge nicely for $E > 0$. This optimism is due to the important fact that the Padé approximant satisfies exact unitarity.[3] For example, the $[N, N]$ Padé approximant to S, which we call $S_{N,N}$, satisfies $S^*_{N,N} = 1/S_{N,N}$. Also, the $[N, M]$ Padé to T, $T_{N,M}$ satisfies $\operatorname{Im} T_{N,M} = |T_{N,M}|^2$ for $M \leq N$.

A proof of the convergence of the Padé is possible for positive energies if we restrict ourselves to positive semi-definite potentials and consider instead the zeros and poles of $\tan \delta$.

Let us define $F(k, g) = f(k, g)/f(k, 0)$ and put

$$\begin{aligned} H(k^2, g) &= (2ikg)^{-1}[F(k, g) - F(-k, g)], \\ G(k^2, g) &= \tfrac{1}{2}[F(k, g) + F(-k, g)]. \end{aligned} \tag{17}$$

Since $F(k, g)$ is an entire function of g of order $\frac{1}{2}$, it follows that H and G are also entire and of order $\leq \frac{1}{2}$. We have seen in Sec. III, however, that for $k = iK$, $0 < K < \mu/2$, the zeros of $F(k, g)$ and $F(-k, g)$ intertwine. This implies that H and G have at least one real zero between two real zeros of F and hence they are precisely of order $\frac{1}{2}$.

From Eqs. (6) and (17) we have

$$(kg)^{-1} \tan \delta\,(k, g) = H(k^2, g)/G(k^2, g). \tag{18}$$

One may note that H and G are real analytic functions of $E = k^2$ for $|\operatorname{Im} k| < \mu/2$. For $E > 0$, the zeros of $H(G)$ are the values of g where the phase shift $\delta(k, g)$ is an even (odd) multiple of $\pi/2$. Denoting the zeros of H and G by $H_n(k^2)$ and $G_n(k^2)$ we may write

$$\begin{aligned} H(k^2, g) &= H(k^2, 0) \prod [1 - g/H_n(k^2)], \\ G(k^2, g) &= \prod [1 - g/G_n(k^2)]. \end{aligned} \tag{19}$$

Because of the threshold behavior of the phase shift, $\delta \sim k^{2l+1}\, a(g)$ for small k, we have $H(k^2, 0) \sim CE^l$. Also, from Eqs. (7) and (17) we have $G_n(0) = g_n(0)$.

We wish to examine the distribution of zeros and poles of $\tan \delta$ and establish the following theorem.

Theorem: If $V(r)$ has no zeros then, apart from subtractions, $(gk)^{-1}E^{-l} \tan \delta$ is a series of Stieltjes in g for $E \geq 0$.

We will establish the theorem by showing that in the formal expansion

$$(kg)^{-1}E^{-l} \tan \delta(k, g) = \sum -\frac{r_n(k^2)/G_n(k^2)}{1 - g/G_n(k^2)}, \tag{20}$$

the residues r_n and poles G_n are real and that $r_n/G_n > 0$. The assumption that $V(r)$ has no zeros is crucial. Without it we cannot even show that $G_n(E)$ and $r_n(E)$ are real for real $E > 0$.

Proof: We first note that the zeros and poles of $\tan \delta$ are real and simple for $E > 0$. This can be shown in the same way as Lemmas 1 and 4 of Sec. II. For example, identity (11) for a general g is

$$\frac{d}{dr} W_r(\phi, \phi^*) + (g - g^*)V(r)\,|\phi|^2 = 0.$$

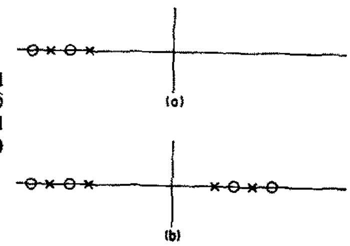


FIG. 3. The zeros 0 and poles X of $(gk)^{-1}E^{-l} \tan \delta$ in the g plane for $E \geq 0$ and $V(r) \geq 0$ for: (a) $E \approx 0$ (b) $E \gg 0$.

For g equal to H_n or G_n and $E > 0$, the Wronskian vanishes both at $r = 0$ and ∞. If $V(r) \geq 0$, one must have H_n and G_n real. That these zeros are simple also follows using a proof analogous to that used in Lemma 4. In fact, writing identity (14) for a general g with $E > 0$ gives us the stronger statement

$$\frac{d\delta}{dg} = -\frac{\pi}{4}\left(\frac{k}{2}\right)^{2l+1} \Gamma^{-2}(l + \tfrac{3}{2})(G^2 + k^2g^2H^2)^{-1} \times \int_0^\infty V(r)\phi^2\, dr. \tag{21}$$

Hence $d\delta/dg < 0$ and δ is a monotonic function of g. It then follows that the zeros and poles of $(kg)^{-1}E^{-l} \tan \delta$ intertwine as in Fig. 3.

One should notice that for sufficiently small energies, there are no zeros or poles for positive g. As E increases new poles and zeros move in from $+\infty$. This behavior is connected with the Wigner inequality[11] on the phase shift. For example, if $V(r)$ is of finite range R, the $l = 0$ phase shift satisfies the inequality

$$\frac{d\delta}{dk} \geq -R + \frac{1}{2k} \sin(2kR + 2\delta) \quad \text{(Ref. 11, Eq. 5.12)},$$

independent of the magnitude of g. If we combine this with the fact that for $g > 0$ and $k = 0$, $d\delta/dk > -R$ [Ref. 11, Eq. (4.32)], we see that $\delta(k, g) \geq -Rk$. Hence for $k < \pi/2R$ the phase shift can never be as negative as $-\pi/2$, there would be no zeros or poles for $g > 0$ and $g^{-1} \tan \delta$ would be an ordinary series of Stieltjes. For a general E, $g^{-1} \tan \delta$ would be an extended series of Stieltjes with a finite number of zeros and poles for positive g. As E tends to ∞ there are no restrictions on the number of zeros and poles for $g > 0$, but because the Born approximation becomes valid all zeros and poles tend to $\pm\infty$.

We may now conclude that if $V(r) \geq 0$ and no subtractions are required in Eq. (20), the $[N, N + j]$ Pade converges to $(kg)^{-1}E^{-l} \tan \delta$ for large N and fixed $j = \pm 1, \pm 3, \cdots$. For sufficiently small energies, one can also include $j = 0, \pm 2, \cdots$.

[11] R. G. Newton, J. Math. Phys. 1, 319 (1960).

V. THE MAXIMUM MODULUS

We wish to examine the Jost function for fixed l and k and large $|g|$ in order to establish its order and type as an entire function of g. The WKBJ or phase-integral method is well suited for this purpose.[12] Although it is not necessary, we restrict ourselves to potentials $V(r) \geq 0$. This simplifies the calculation since there is no need to take into account turning points.

The order and type of an entire function[13] $f(z)$ classify the rate of growth of the function for large $|z|$. Let $M(|z|)$ be the maximum value of $|f(z)|$ on the circle of radius $|z|$. The order ρ and type σ of $f(z)$ are defined as

$$\rho = \limsup_{r\to\infty} \frac{\ln \ln M(r)}{\ln r},$$
$$\sigma = \limsup_{r\to\infty} r^{-\rho} \ln M(r).$$

Roughly speaking, if $f(z)$ is of type ρ and order σ it has the same maximum rate of growth as the function $\exp(\sigma z^\rho)$.

Let us now use the Schrödinger equation to find the asymptotic form of the Jost function for large $|g|$. In Eq. (1) we consider $k^2 = -K^2$ with $K > 0$ and make the changes of variables $r = e^x$, $y = e^{x/2}\omega$ to obtain

$$\frac{d^2\omega}{dx^2} - q^2(x)\omega = 0,$$
$$q(x) = [K^2e^{2x} + gV(e^x)e^{2x} + (l + \tfrac{1}{2})^2]^{\frac{1}{2}}. \tag{22}$$

For large positive values of g this has the approximate solution

$$\omega = q^{-\frac{1}{2}} C \exp\left(\pm \int q\, dx\right). \tag{23}$$

For the regular solution ϕ, Eq. (23) becomes in terms of r

$$\phi = C[p(r)]^{-\frac{1}{2}} \exp \int_a^r p(t)\, dt, \tag{24}$$

where

$$p(r) = [K^2 + gV(r) + (l + \tfrac{1}{2})^2/r^2]^{\frac{1}{2}} \tag{25}$$

and

$$C = (l + \tfrac{1}{2})^{\frac{1}{2}} a^{l+\frac{1}{2}} \exp \int_0^a \{p(r) - [(l + \tfrac{1}{2})/r]\}\, dr. \tag{26}$$

If the asymptotic form of Eq. (24) is compared with Eq. (5), one obtains the Jost function

$$f(-iK, g) = 2a^{l+\frac{1}{2}}[K(l + \tfrac{1}{2})]^{\frac{1}{2}} \times \exp\left\{\int_0^a [p(r) - ((l + \tfrac{1}{2})/r) - K]\, dr + \int_a^\infty [p(r) - K]\, dr\right\} \tag{27}$$

for $g \gg 0$ and $K > 0$. From Eq. (27) we have

$$\lim_{g\to\infty} \ln f(-iK, g) = g^{\frac{1}{2}} \int_0^\infty [V(r)]^{\frac{1}{2}}\, dr. \tag{28}$$

Since Eq. (28) is also valid for complex g with $-\pi < \arg g < \pi$, we conclude that the Jost function has $\rho = \frac{1}{2}$ and $\sigma = \int_0^\infty [V(r)]^{\frac{1}{2}}\, dr$.

VI. DISCUSSION

We have obtained the general features of the partial-wave S-matrix element $S(k, g)$ as a function of the potential strength g for a large class of potentials. Because $S(k, g)$ is related to a series of Stieltjes in g, we were able to prove the convergence of the Padé method of summing the Born series. For example, in the most optimistic case of one subtraction in Eq. (10) and no subtractions in Eq. (20), the results for $E < 0$ and $E \geq 0$ can be combined to state that, for positive semi-definite Yukawa type potentials, the $[N, N]$ Padé converges to $S(k, g)$ for large N and $-\mu^2/4 < E$. For finite N, the Padé yields approximate expressions for $S(k, g)$, which are good outside the usual radius of convergence of the perturbation expansion. These approximate expressions have the additional features of satisfying exact unitarity and supplying rigorous bounds[2] on $S(k, g)$ below threshold and $\tan\delta$ above threshold. Since they are valid outside the radius of convergence they may be used to obtain the bound states and resonances from the Born series itself.

Numerically, the rate of convergence appears to be quite rapid. For example, for the Yukawa potential ge^{-r}/r the s-wave scattering length is $a_s = -g + (g^2/2) - g^3 \ln \frac{4}{3} + g^4 \ln \frac{32}{27}$. The $[1, 1]$ Padé to a_s predicts an $E = 0$ bound state at $g = -2$, the $[2, 2]$ Padé at $g = -1.684$. The exact value to 4 figures is $g = -1.680$. Also for positive g the Padé supplies the bounds $[N, N] \geq a_s \geq [N, N + 1]$. For the Hulthen potential[14] $g(e^r - 1)^{-1}$ for $g = 1$ (the Born series is divergent), this gives tor $N = 1$, $-1.29 \geq a_s \geq -1.35$. The convergence appears to be less rapid at larger energies and/or larger angular momentum.

The eventual hope of course is that the Padé approximant may be successfully applied to the field theory perturbation expansions of strong interactions. Preliminary results on πN and $\pi\pi$ scattering look encouraging.[15]

---

[12] J. Heading, *An Introduction to Phase Integral Methods* (Methuen and Company, Ltd., London, 1962).

[13] R. H. Boas, Jr., *Entire Functions* (Academic Press Inc., New York, 1954).

[14] S. Weinberg, J. Math. Phys. **5**, 743 (1964).

[15] J. J. Kubis, Bull. Am. Phys. Soc. **12**, 472 (1967); M. A. Newton (private communication); D. Bessis and M. Pusterla, Phys. Letters **25B**, 279 (1967); L. A. Copley and D. Masson, Phys. Rev. (to be published).